Stabilization of linear finite-dimensional systems using continuous neural networks

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Abstract— **We investigate a class of linear finite-dimensional systems and establish their stability via the design of a feedback law constructed from a continuous neural network. The main novelty of this work lies in the fact that the considered neural network consists of infinite number of neurons, continuously distributed within each hidden layer. To appropriately bound the nonlinearity induced by the activation functions and guarantee stability, a suitable sector constraint is introduced. A notable advantage of the proposed approach is that the obtained assumptions remain linear, even though the resulting closed-loop system is nonlinear. Furthermore, we emphasize the effect of considering an infinite number of neurons, thereby demonstrating that the present work extends several existing results. Numerical simulations are provided to illustrate that increasing the network width leads to a larger region of attraction and owns more robustness.**

I. INTRODUCTION

The stabilization of linear systems using neural network (NN) feedback controllers can be viewed as an extension of the well-addressed problem of stabilization with saturated actuators, which has attracted significant interest over the past decades [1], [7], [12], [19]. This interest is motivated by the fact that, in many control applications, input signals are inherently limited in amplitude due to physical, safety, or technological constraints. So, neglecting these amplitude limitations in the stability analysis of control systems may lead to degraded performance or even instability [3]. Stabilization using NN controllers naturally accounts for such amplitude limitations and also enhances robustness with respect to uncertainties in the system.

Despite the numerous contributions to feedback stabilization with saturated controls, the general framework for designing neural network feedback controllers has received comparatively limited attention. This is mainly due to the structural complexity of NNs, which may involve various types of nonlinear activation functions, multiple layers, and a large number of neurons. Among the relatively limited existing literature, one may cite for instance [20], where the authors propose a feedback stabilization result for a plant system interconnected with a NN by using quadratic constraints (QCs) to bound the nonlinear activation functions. The QC framework has also been employed in [5] to compute outer bounds on the outputs of a static NN given a set of inputs, and in [6], [17] to upper-bound the Lipschitz constant of NNs. In [11], QCs are constructed from bounds on the partial gradients of NN controllers in order to guarantee stability. Furthermore, the work in [14] establishes global asymptotic stability for dynamic NN models using QCs combined with Lyapunov theory. In [9], both positivity and global exponential stability are stated using QCs and the Lur'e system method with the Aizerman conjecture.

However, in all the aforementioned works, the NNs under consideration consist of a finite number of neurons. In contrast, the recent work [18] introduces a width continuous representation of NNs with an infinite number of neurons (see also [15]). It turns out that this continuous NN is more representative since it can be interpreted as the limiting case of classical neural networks. NNs with infinite number of neurons arise naturally when considering continuum approximations of brain structure by the high density of neurons [4]. From the perspective of control applications, neural networks with a large number of neurons show significant promise for addressing complex problems in nonlinear control, such as the feedback control of tokamak plasmas [2]. In this paper, we consider a NN with an infinite number of neurons and employ it as controller for the feedback stabilization of finite-dimensional linear systems. More precisely, the contributions of this paper are as follows.

- We establish stability results for linear ordinary differential equations and discrete-time plant systems by constructing a nonlinear feedback law based on a continuous NN. Since classical NNs are the approximation of continuous NNs [15], [18], our results extend existing approaches and also provide improved robustness with respect to approximation errors;
- Extending the QC-based existing approaches, we introduce a $W-$sector condition to bound the nonlinear activation functions, from which we derive tractable LMI conditions ensuring the stability of the resulting nonlinear closed-loop system. This framework also enables the approximation of the region of attraction (ROA). In particular, we show that this region can be enlarged by increasing the number of layers in the continuous NN;
- The obtained stability results indicate that continuous NN controllers not only reduce the initial transient overshoot but also exhibit greater robustness to uncertainties (on numerical simulations) than linear feedback laws.

The paper is organized as follows. Section II introduces

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the continuous NN and preliminaries. Section III presents the main results. In Section IV, we illustrate the proposed theorems. The proofs are given in Section V and conclusions in Section VI.

Notation: Let $\mathscr{X} := C^0([0,1],\mathbb{R})$ and for any $\ell \in \mathbb{N}^*$, $\mathscr{X}^\ell$ denotes the ℓ-fold Cartesian product of $\mathscr{X}$. For $x, y \in \mathbb{R}^\ell$, the relation $x \leq y$ is understood component-wise, i.e., $x_i \leq y_i$ for all $i \in \{1,\dots,\ell\}$. Similarly for $\underline{z}, \bar{z} \in \mathscr{X}^\ell$, $\underline{z} \leq \bar{z}$ means $\underline{z}_i(\tau) \leq \bar{z}_i(\tau)$ for all $\tau \in [0,1]$ and for all $i \in \{1,\dots,\ell\}$. For $z \in \mathscr{X}^\ell$, $z \geq 0$ means $z_i(\tau) \geq 0$ for all $\tau \in [0,1]$ and for all $i \in \{1,\dots,\ell\}$. For any $\underline{v}, \bar{v} \in \mathscr{X}^\ell$, we denote $\mathscr{I}(\underline{v},\bar{v}) := \{v \in \mathscr{X}^\ell,\ \underline{v}(\tau) \leq v(\tau) \leq \bar{v}(\tau),\ \forall \tau \in [0,1]\}$. For $\varphi \in C^0(\mathbb{R})$ and $v \in \mathscr{X}$, $\varphi(v)$ corresponds to $\varphi \circ v$ and if $v = (v_1,\dots,v_\ell) \in \mathscr{X}^\ell$, $\varphi(v) = (\varphi(v_1),\dots,\varphi(v_\ell))$. For a given symmetric matrix $Q \in \mathbb{R}^{n\times n}$, $Q \succ 0$, (resp. $Q \prec 0$) means it is positive-definite (resp. negative-definite) and $Q \succeq 0$ means it is positive semi-definite. For a given vector $x_* \in \mathbb{R}^n$, we define $\Pi(Q,x_*) := \{x \in \mathbb{R}^n : (x-x_*)^\top Q(x-x_*) \leq 1\}$. The symbol $\star$ in a given matrix stands for symmetric block.

II. PROBLEM STATEMENT

A. Continuous neural network

We consider a feed-forward neural network π_c without bias, with ℓ hidden layers, where each layer contains a continuum of neurons indexed by $\tau \in [0,1]$. This network is known as continuous neural network [18], and is described as follows: for a given input $x \in \mathbb{R}^n$, for all $\tau \in [0,1]$,

$$z_0(\tau) = L(\tau)x \tag{1a}$$

$$z_i(\tau) = \phi_i\left(\int_0^1 w_i(\tau,s)z_{i-1}(s)ds\right),\ \forall i \in \{1,\dots,\ell\} \tag{1b}$$

$$u = \int_0^1 w_{\ell+1}(s)z_\ell(s)ds, \tag{1c}$$

where the lifting function (also called the input-to-hidden function) $L : [0,1] \to \mathbb{R}^{1\times n}$ is continuous. For every $i \in \{1,\dots,\ell\}$, the kernel $w_i : [0,1]^2 \to \mathbb{R}$ is also continuous and corresponds to the hyper-parameter function of the i-th layer. The output weight function $w_{\ell+1} : [0,1] \to \mathbb{R}^p$ is a vector-valued function so that the integral in (1c) is understood component-wise. The activation mapping $\phi = (\phi_1,\dots,\phi_\ell) : \mathbb{R} \to \mathbb{R}^\ell$ is assumed to be locally Lipschitz and is often selected as the saturation map, the hyperbolic tangent function, ReLU or the sigmoid. Given an input x, the continuous NN defined by (1) admits a unique continuous solution $(z_0,\dots,z_\ell,u)$. The continuous NN defined in (1) differs fundamentally from the deep NN studied in [5], [6], [9], [20] since in each of its layers there is an infinite number of neurons indexed over $[0,1]$. It is shown in [18] that the continuous NN formulation (1) can be seen as an extension of the traditional feed-forward neural network, which contains finite number of neurons inside each layer. More precisely, a continuous NN arises as the limit of a deep NN when the number of neurons per layer tends to infinity.

Let us first introduce the following NN input maps that will be used throughout the paper. Define $W_i : \mathscr{X} \to \mathscr{X}$, for $i \in \{1,\dots,\ell\}$, and $W_{\ell+1} : \mathscr{X} \to \mathbb{R}^p$ by, for any $z \in \mathscr{X}$,

$$W_i z(\tau) := \int_0^1 w_i(\tau,s)z(s)ds, \quad \forall \tau \in [0,1], \tag{2}$$

$$W_{\ell+1} z := \int_0^1 w_{\ell+1}(s)z(s)ds. \tag{3}$$

In particular, when W_i acts on a vector-valued function, it is understood component-wise. Observe that, for each $i \in \{1,\dots,\ell\}$, the operator W_i maps functions to functions, whereas $W_{\ell+1} = (W^1_{\ell+1},\dots,W^p_{\ell+1})$ maps functions z to finite-dimensional vectors with components $W^i_{\ell+1}z$. This is consistent with the fact that the continuous NN (1) produces a finite-dimensional output u. For every $i \in \{1,\dots,\ell\}$, define

$$v_i : [0,1] \ni \tau \longmapsto \int_0^1 w_i(\tau,s)z_{i-1}(s)ds \in \mathbb{R} \tag{4}$$

where z_i is given by (1a)-(1b). So, given x, it can be propagated through the continuous NN π_c defined in (1) to get the associated quantities v_i and z_i for each layer $i \in \{1,\dots,\ell\}$. This yields the stacked vectors $v = (v_1,\dots,v_\ell)^\top$ and $z = (z_1,\dots,z_\ell)^\top$ satisfying $z = \phi(v)$.

To bound the nonlinearity arising from the activation functions ϕ, we introduce the following sector condition.

Definition 1 (**$W-$sector condition**)**:** Let $\underline{\nu}, \overline{\nu}, \nu_* \in \mathscr{X}$, $\alpha, \beta, \in \mathbb{R}$ with $\underline{\nu} \leq \overline{\nu}$, $\alpha \leq \beta$ and $\nu_* \in \mathscr{I}(\underline{\nu},\overline{\nu})$. Let $W : \mathscr{X} \to \mathbb{R}$ be a continuous linear operator. The function $\varphi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $W-$*offset sector condition* $[\alpha,\beta]$ around the function ν_* on $\mathscr{I}(\underline{\nu},\overline{\nu})$, if for every $\nu \in \mathscr{I}(\underline{\nu},\overline{\nu})$,

$$W(\delta\varphi(\nu) - \alpha\delta\nu)W(\beta\delta\nu - \delta\varphi(\nu)) \geq 0 \tag{5}$$

where $\delta\varphi(\nu) := \varphi(\nu) - \varphi(\nu_*)$, and $\delta\nu := \nu - \nu_*$. ∘

In the case $\nu_* \equiv 0$, φ is said to satisfy a $W-$ sector condition. For a $p-$valued operator $W = (W^1,\dots,W^p) : \mathscr{X} \to \mathbb{R}^p$, the $W-$offset sector condition is satisfied if each $W^i\varphi$ fulfills the above property.

In comparison, the classical sector condition requires

$$(\delta\varphi(\xi) - \alpha\delta\xi)(\beta\delta\xi - \delta\varphi(\xi)) \geq 0, \quad \forall \xi \in [\underline{\xi},\overline{\xi}] \subset \mathbb{R},$$

where $\delta\varphi(\xi) := \varphi(\xi) - \varphi(\xi_*)$, $\delta\xi := \xi - \xi_*$ for a given $\xi_* \in [\underline{\xi},\overline{\xi}]$. Definition 1 can thus be interpreted as an extension of the classical sector condition [19], [20] to a framework involving an infinite number of neurons. For example, by setting $W_0 z := \int_0^1 z(\tau)d\tau$, and considering $\nu_* \equiv 0$, the W_0-sector condition for a given function φ can be geometrically interpreted in the $(a,b)-$plane, where

$$a = \int_0^1 \nu(\tau)d\tau, \quad b = \int_0^1 \varphi(\nu(\tau))d\tau. \tag{6}$$

Each admissible function $\nu \in \mathscr{I}(\underline{\nu},\overline{\nu})$ is mapped to a point $(a,b) \in \mathbb{R}^2$. Condition (5) requires that this set of points lies within the sector delimited by the lines $b = \alpha a$ and $b = \beta a$ as illustrated in Figure 1. For instance, the saturation function $\varphi(\xi) = \mathrm{sat}(\xi) = \max\{-1,\min\{1,\xi\}\}$ and ReLU $\varphi(\xi) = \max\{0,\xi\}$ satisfy this W_0-sector condition with appropriate bounds, giving examples of functions φ fulfilling Definition 1. In the following, $W_{\ell+1} \circ \ldots \circ W_j$ denotes the

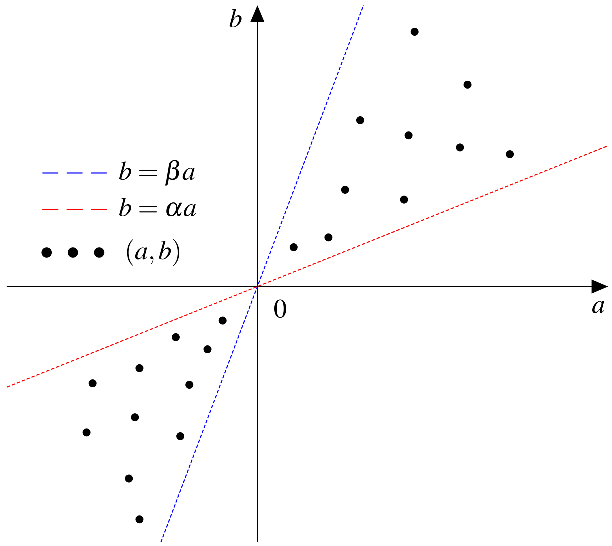

Fig. 1. Schematic representation of the W_0-sector condition for several v

composition of the maps W_i from $\ell+1$ to j (so that if $j=\ell+1$ it reduces to $W_{\ell+1}$). Now, we make the following assumption on the continuous NN π_c.

Assumption 1: *Let $x_* \in \mathbb{R}^n$, and $v_* = (v_{*1},\dots,v_{*\ell})^\top \in \mathscr{X}^l$ the corresponding input function from the continuous NN π_c given by* (1). *Let $\underline{v}_1, \overline{v}_1 \in \mathscr{X}$ with $\underline{v}_1 \le \overline{v}_1$, $v_{*1} = (\underline{v}_1+\overline{v}_1)/2 \in \mathscr{I}(\underline{v}_1,\overline{v}_1)$ and their corresponding bound $\underline{v} = (\underline{v}_1,\dots,\underline{v}_\ell)^\top$, $\overline{v} = (\overline{v}_1,\dots,\overline{v}_\ell) \in \mathscr{X}^\ell$. Assume there exist $\alpha_\phi = (\alpha_1,\dots,\alpha_\ell)^\top$, $\beta_\phi = (\beta_1,\dots,\beta_\ell)^\top \in \mathbb{R}^\ell$ with $\alpha_i \le \beta_i$ such that for any $i \in \{1,\dots,\ell\}$, ϕ_i in* (1) *satisfies the $W_{\ell+1}\circ\ldots\circ W_{i+1}-$offset sector condition $[\alpha_i,\beta_i]$ around the function v_{*i} on $\mathscr{I}(\underline{v}_i,\overline{v}_i)$, where W_i is given in* (2)-(3).

Remark 1 (Computation of the bounds): The computation of the bounds $\underline{v}$ and $\overline{v}$ on the input v of activation function ϕ in Assumption 1 is performed following the procedures described in [8], [20]. Roughly speaking, we start with functions $\underline{v}_1$ and $\overline{v}_1$ such that $v_{*1} \in \mathscr{I}(\underline{v}_1,\overline{v}_1)$. And, using the continuous NN (1) these bounds can be propagated through the network: the first layer yields bounds $\mathscr{I}(\underline{z}_1,\overline{z}_1)$ for the output $z_1=\phi_1(v_1)$ which can be used to deduce bounds $\mathscr{I}(\underline{v}_2,\overline{v}_2)$ of v_2 and so on for subsequent layers. These bounds always contain the corresponding z_{*i} and v_{*i}. By iterating this procedure through all layers, one can generate bounds $\underline{v},\overline{v}$ for v. This means that, for a given $x\in\mathbb{R}^n$, if the corresponding v_1 is such that $v_1 \in \mathscr{I}(\underline{v}_1,\overline{v}_1)$, then by construction of $\underline{v}_i,\overline{v}_i$, for any $i\in\{1,\dots,\ell\}$, the corresponding $v_i \in \mathscr{I}(\underline{v}_i,\overline{v}_i)$. ◦

B. Control system

We consider the linear ordinary differential equation

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{7}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^p$, stands for the state and the input respectively, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times p}$ and $x_0\in\mathbb{R}^n$ is the initial condition. The primary objective of this paper is to guarantee the stability of system (7) by designing a feedback control as the output of the continuous NN π_c given by (1). In other words, the goal is to generate u from the NN π_c as

$$u(x) := \pi_c(x) = W_{\ell+1}\phi_\ell\left(W_\ell\phi_{\ell-1}(\ldots W_2\phi_1(W_1Lx)\ldots)\right), \tag{8}$$

to ensure the stability of (7)-(8). The feedback law π_c is nonlinear whenever at least one of the ϕ_i is nonlinear, and it reduces to a linear feedback when each ϕ_i is linear. This highlights that the neural-network-based feedback design framework not only enables the stabilization of nonlinear closed-loop systems, but also remains consistent with the classical construction of linear feedback laws for finite-dimensional systems. Regarding this point, the reader may wonder why the nonlinear feedback u in (8) is considered, while the classical pole-shifting method could be used to design a linear feedback ensuring global stabilization of (7). The reasons are the following: the nonlinear feedback controls u obtained from the continuous NN

- are more robust to uncertainties than linear feedback laws derived from pole shifting in certain situations;
- significantly reduce the initial transient overshoot compared to linear feedback laws;
- are more representative of several control applications, particularly when the control input is subject to amplitude limitations.

We define an equilibrium point for the closed-loop (7)-(8).

Definition 2 (Equilibrium point): A point $(x_*,u_*) \in \mathbb{R}^n\times\mathbb{R}^p$ is said to be an equilibrium for (7)-(8), if

$$Ax_* + Bu_* = 0, \tag{9}$$

and $u_* = \pi_c(x_*)$. ◦

III. MAIN RESULTS

We introduce first some notation to facilitate the presentation of the main results. We define for all $i,k\in\{1,\dots,\ell\}$,

$$u_i^k := \prod_{j=i}^{k}\alpha_j, \;\; \theta_i := \beta_i - \alpha_i, \tag{10}$$

where α,β are given in Assumption 1. We also adopt the conventions

$$u_0^k = u_j^0 = u_{k+j}^k = u_j^{\ell+j} = 1 \;\forall k,j\in\mathbb{N}^*.$$

For a given $\lambda = (\lambda_1,\dots,\lambda_\ell)\in\mathbb{R}^{p\ell}$ with $\lambda_i\in\mathbb{R}^p$ and $\lambda_i>0$, we define the matrices $\Lambda_i\in\mathbb{R}^{p\times p}$, $\mathscr{K},K\in\mathbb{R}^{p\times n}$, as

$$\Lambda_i := \mathrm{diag}(\lambda_i),\; K := W_{\ell+1}\circ\ldots\circ W_1L, \quad \mathscr{K} := u_1^\ell K, \tag{11}$$

and for a symmetric positive definite matrix $Q\in\mathbb{R}^{n\times n}$,

$$M(\lambda,K,Q) := \left(u_{i+1}^\ell B^\top Q + \theta_i u_1^{i-1}\Lambda_i K\right)^\top_{1\le i\le\ell} \tag{12}$$

and $T(\lambda)$ as

$$T(\lambda) := \begin{bmatrix} -2\Lambda_1 & \star & \dots & \star & \star \\ \theta_2u_2^1\Lambda_2 & -2\Lambda_2 & \dots & \star & \star \\ \theta_3u_2^2\Lambda_3 & \theta_3u_3^2\Lambda_3 & \dots & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{\ell-1}u_2^{\ell-2}\Lambda_{\ell-1} & \theta_{\ell-1}u_3^{\ell-2}\Lambda_{\ell-1} & \dots & -2\Lambda_{\ell-1} & \star \\ \theta_\ell u_2^{\ell-1}\Lambda_\ell & \theta_\ell u_3^{\ell-1}\Lambda_\ell & \dots & \theta_\ell u_\ell^{\ell-1}\Lambda_\ell & -2\Lambda_\ell \end{bmatrix}. \tag{13}$$

We present now the stability result for the closed-loop system (7)–(8). The proof is postponed to Section V-A.

Theorem 1: *Consider the system* (7) *in closed-loop with π_c under Assumption 1. If there exist a symmetric matrix $Q \in \mathbb{R}^{n\times n}$ with $Q \succ 0$ and a vector $\lambda \in \mathbb{R}^{p\ell}$, with $\lambda > 0$, such that*

$$\begin{bmatrix} (A+B\mathscr{K})^\top Q + Q(A+B\mathscr{K}) & \star \\ M(\lambda,K,Q) & T(\lambda) \end{bmatrix} \prec 0, \tag{14}$$

$$\begin{bmatrix} (\overline{v}_1(\tau)-v_{*1}(\tau))^2 & \star \\ (W_1L(\tau))^\top & Q \end{bmatrix} \succeq 0, \quad \forall \tau \in [0,1], \tag{15}$$

then the closed-loop system (7)-(8) *is locally exponentially stable around x_*. In particular, the set $\Pi(Q,x_*)$ is included in the region of attraction (ROA).*

From the expressions (12) and (13), it follows that the matrices appearing in (14), (15) depend linearly on Q and λ. In other words, (14) and (15) are linear matrix inequalities (LMIs) and resemble the LMIs proposed in [19] when dealing with a nested saturated system. Therefore, the search for these unknown variables can be efficiently carried out using standard optimization solvers such as SDPT3 or MOSEK. Furthermore, the LMI (14) can be interpreted as a generalization of the classical linear state-feedback stabilizability condition. Indeed, if the activation function ϕ of the continuous NN π_c is linear, then Assumption 1 is satisfied globally with $\alpha = \beta$ and in this case, a straightforward computation shows that (14) is equivalent to $(A+B\mathscr{K})^\top Q + Q(A+B\mathscr{K}) \prec 0$.

To further exhibit the tractability of the LMI (14), let us consider the particular case where the continuous NN π_c consists of a single layer, that is, $\ell = 1$. In this case, the Assumption 1 reduces to a simple sector condition on the activation function ϕ_1. The corresponding result is stated in the following corollary. Its proof is omitted, as it follows directly from Theorem 1.

Corollary 1 (With $\ell = 1$): Consider the system (7) with the continuous NN π_c given by (1) with $\ell = 1$, and let (x_*, u_*) be an equilibrium with corresponding v_{*1}, z_{*1}. Assume that there exists $\alpha_1, \beta_1 \in \mathbb{R}^p$ with $\alpha_1 \le \beta_1$ and $\underline{v}_1 \le \overline{v}_1 \in \mathscr{X}$ such that ϕ_1 satisfies the W_2-offset sector $[\alpha_1, \beta_1]$ on $v_{*1} := (\underline{v}_1 + \overline{v}_1)/2 \in \mathscr{I}(\underline{v}_1, \overline{v}_1)$. If there exist a symmetric matrix $Q \in \mathbb{R}^{n\times n}$ with $Q \succ 0$ and a vector $\lambda_1 \in \mathbb{R}^p$ with $\lambda_1 > 0$ such that (15) holds and

$$\begin{bmatrix} (A+B\mathscr{K})^\top Q + Q(A+B\mathscr{K}) & \star \\ B^\top Q + \theta_1 \Lambda_1 K & -2\Lambda_1 \end{bmatrix} \prec 0, \tag{16}$$

where $K = W_2 W_1 L$, $\mathscr{K} = \alpha_1 K$, then the closed-loop system (7)-(8) is locally exponentially stable around x_* and the set $\Pi(Q,x_*)$ is included in the ROA. ◇

Observe that the LMI (16) in Corollary 1 involves a $(n+p)\times(n+p)$ matrix. In contrast, Theorem 1 requires an LMI of dimension $(n+p\ell)\times(n+p\ell)$. This shows that using a small number of layers in the continuous NN reduces the size of the matrix in the LMI, thereby facilitating its numerical tractability. However, we should point out that the size of the matrix $T(\lambda)$ is $(p\ell, p\ell)$, which increases proportionally to ℓ. Consequently, when ℓ is large, we have more degrees of freedom to verify the LMI (14). It follows that a small number of layers often leads to a more restrictive ROA, whereas adding layers may enlarge the ROA within which stability is guaranteed. This is clearly illustrated in Section IV.

The assumptions of Theorem 1 can be adapted to address the stability of the discrete-time linear system considered in [20], which is modeled as

$$x[k+1] = Ax[k] + Bu[k], \tag{17}$$

where $x[k] \in \mathbb{R}^n$ is the state, $u[k] \in \mathbb{R}^p$ is the control input, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times p}$. We denote by $\tilde{M}$ the adaptation of M defined in (12), $\tilde{M}(\lambda,K,Q) := \left(u_{i+1}^{\ell} B^\top QA + \theta_i u_1^{i-1}\Lambda_i K\right)^\top_{1\le i\le \ell}$ and we define $\mathbb{B}(Q) := (u_{i+1}^{\ell} u_{j+1}^{\ell} B^\top QB)_{1\le i,j\le \ell}$ as the block matrix whose (i,j)-th entry is the $p\times p$ matrix $u_{i+1}^{\ell} u_{j+1}^{\ell} B^\top QB$. We now state the corresponding stability result for the plant (17) controlled by the continuous NN π_c defined in (1). The proof is omitted, as it follows along the similar reasoning as that of Theorem 1.

Theorem 2: *Consider the system* (8)-(17) *with an equilibrium (x_*, u_*), i.e., satisfying $Ax_* + Bu_* = x_*$ and $u_* = \pi_c(x_*)$. Suppose that Assumption 1 holds true. If there exist a symmetric matrix $Q \in \mathbb{R}^{n\times n}$ with $Q \succ 0$ and a vector $\lambda \in \mathbb{R}^{p\ell}$ with $\lambda > 0$ such that* (15) *holds and*

$$\begin{bmatrix} (A+B\mathscr{K})^\top Q(A+B\mathscr{K}) - Q & \star \\ \tilde{M}(\lambda,K,Q) & \mathbb{B}(Q)+T(\lambda) \end{bmatrix} \prec 0,$$

then the equilibrium x_* is locally exponentially stable and in particular, $\Pi(Q,x_*)$ is included in the ROA.

Remark 2 (Theorem 2 vs [20]): Theorem 2 extends the stability result established in [20], where the authors consider a neural network with a finite number of neurons. Indeed, according to [18], for any x belonging to a given compact set and for any $\varepsilon > 0$, there exist $r > 0$, matrices $\mathscr{L} \in \mathbb{R}^{n\times n}$, $\mathscr{W}_i \in \mathbb{R}^{n\times n}$, $\mathscr{W}_{\ell+1} \in \mathbb{R}^{p\times n}$, and vectors $X, Z_i \in \mathbb{R}^n$, $U \in \mathbb{R}^p$ satisfying

$$\pi : \begin{cases} Z_0 = \mathscr{L}X, \\ Z_i = \phi_i(\mathscr{W}_i Z_{i-1}), \quad \forall i \in \{1,\dots,\ell\} \\ U = \mathscr{W}_{\ell+1} Z_\ell, \end{cases} \tag{18}$$

and $\|x - X\|_{\mathbb{R}^n} \le r$ implies $\|u - U\|_{\mathbb{R}^p} \le \varepsilon$. In other words, the continuous NN π_c defined in (1) can be approximated by a neural network π, and there exists an error function e such that $u = \pi_c(x) = \pi(x) + e(x)$ for all x in a given compact set. The representation π in (18) corresponds to a deep NN with ℓ hidden layers and n neurons per layer. Theorem 2 establishes the stability of the closed-loop system $x[k+1] = Ax[k] + B\pi(x[k]) + Be(x[k])$. This differs from the stability of $x[k+1] = Ax[k] + B\pi(x[k])$, which is stated in [20, Theorem 1]. Both are equal if and only if $e \equiv 0$. Consequently, the stability result we have obtained using the continuous NN is more robust, as it explicitly accounts for approximation errors. ◦

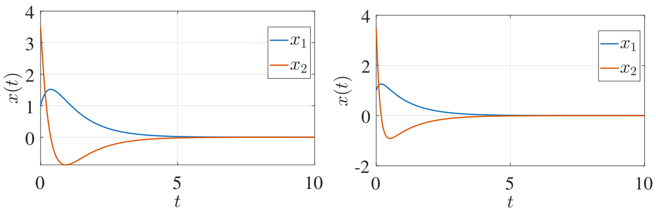

Fig. 2. Solutions of (19) left ($\ell = 1$): u is (22), right ($\ell = 2$): u is (24)

IV. ILLUSTRATIVE EXAMPLES

We now illustrate the applicability of Theorem 1 through an academic example. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u. \tag{19}$$

We consider a continuous NN π_c of the form (1) with parameters given by $L(\tau) = (\tau, \tau)$ and $w_1(\tau, s) = \tau s$. We select $x_* = (0, 0)$, $u_* = 0$ and notice that the corresponding $v_* = z_* = 0$. For any $\xi \in \mathbb{R}$, $\phi_1(\xi) = \text{sat}(\xi) = \max\{-1, \min\{1, \xi\}\}$. The choice of the parameters w_i and L in this section is made solely to simplify the integration and to ensure that the operator $A + B\mathscr{K}$ is Hurwitz, as required by the LMI (14).

First case (With $\ell = 1$): We set here $w_2(s) = 24$ for all $s \in [0, 1]$. The function ϕ_1 satisfies the W_2- sector condition $[\alpha_1, \beta_1]$ around the function $v_{*1} \equiv 0$ on $\mathscr{I}(\underline{v}_1, \overline{v}_1)$ where $\alpha_1 = 1/2$, $\beta_1 = 2$ and $\underline{v}_1 \equiv -1$, $\overline{v}_1 \equiv 1$. Indeed, for any $v \in \mathscr{I}(\underline{v}_1, \overline{v}_1)$, $\phi_1(v) = v$ and

$$\begin{aligned} &W_2(\delta\phi_1(v) - \alpha_1 \delta v) W_2(\beta_1 \delta v - \delta\phi_1(v)) \\ &= (1 - \alpha_1)(\beta_1 - 1)\left(24 \int_0^1 v(\tau) d\tau\right)^2 \geq 0. \end{aligned} \tag{20}$$

A simple computation with (10), (11), yields $\theta_1 = 3/2$, $u_1^1 = 1/2$, $K = W_2 W_1 L = (4, 4)$ and $\mathscr{K} = (2, 2)$. Using the SDPT3 solver in MATLAB and minimizing the trace of Q we get by searching $\lambda_1 \in [0.5, 1]$, the following optimal values

$$\lambda_1^{opt} = 0.5, \quad Q_1^{opt} = \begin{bmatrix} 1.6 & 1.4 \\ 1.4 & 1.5 \end{bmatrix}. \tag{21}$$

Therefore, applying Corollary 1, we get the following result.

Proposition 1: *The origin of the system* (19) *is locally exponentially stable with the nonlinear feedback u :*

$$u(x) = 24 \int_0^1 sat\left(\frac{1}{3}\tau(x_1 + x_2)\right) d\tau, \tag{22}$$

and the ROA contains $\Pi_1(Q_1^{opt}, 0)$.

The solution of (19) in closed-loop with (22) with initial condition $x_0 = (1, 3.5)$ is plotted in Figure 2 (left). We also highlight in Figure 5 (left), that the solution of (19)-(22) diverges when considering the initial condition $x_0 = (11, 11.4)$, which is outside of the ROA.

Second case (With $\ell = 2$): We consider the same parameters $\alpha_1, \beta_1, v_1, \phi_1$ as in the case $\ell = 1$. We set $w_2(\tau, s) = \tau s$, $w_3(s) = 72$ for all $\tau, s \in [0, 1]$ and $\phi_2(\xi) = \max\{-1, \min\{1, 2\xi\}\}$. With $\alpha_2 = 1, \beta_2 = 3$, a straightforward computation shows that ϕ_1 satisfies the $W_3 \circ W_2-$sector condition $[\alpha_1, \beta_1]$ and ϕ_2 satisfies the W_3-sector condition $[\alpha_2, \beta_2]$. Moreover we still have $K = (4, 4)$ and $\mathscr{K} = (2, 2)$. Solving the minimization problem of the trace of Q under constraints (14), (15), by SDPT3 solver in MATLAB we get by searching $\lambda_i \leq 1$, that

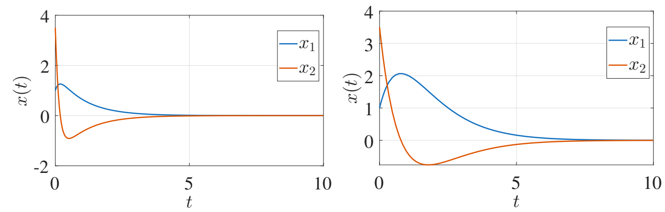

Fig. 3. Solutions of (19) left ($\ell = 2$): u is (24), right (linear): $u = \mathscr{K}x$.

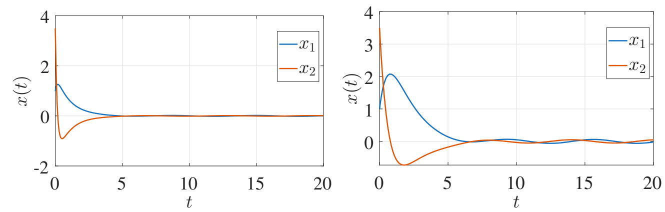

Fig. 4. Solutions when adding uncertainty, left: u is (24), right: $u = \mathscr{K}x$.

$$\lambda_1^{opt} = 0.25, \quad \lambda_2^{opt} = 0.1, \qquad Q_2^{opt} = \begin{bmatrix} 1.01 & 0.8 \\ 0.8 & 0.81 \end{bmatrix}. \tag{23}$$

In view of (21) and (23), $\det(Q_2^{opt}) < \det(Q_1^{opt})$. Noticing that the volume of the ellipsoid $\{x \in \mathbb{R}^n : x^\top Q x \leq 1\}$ is inversely proportional to the determinant of Q, we conclude that a larger ROA is obtained with $\ell = 2$. This is plotted in Figure 6 (right) where the ellipsoid with Q_2^{opt} strictly includes the one with Q_1^{opt}. Now, applying Theorem 1, we get this result.

Proposition 2: *The origin of* (19) *is locally exponentially stable with the feedback law*

$$u(x) = 72 \int_0^1 sat\left(2\tau \int_0^1 sat\left(\frac{1}{3}s(x_1 + x_2)\right) sds\right) d\tau \tag{24}$$

and $\Pi(Q_2^{opt}, 0)$ is included in the ROA.

The solution of (19)-(24) is plotted in Figure 2 (right). Moreover, in Figure 5, we exhibit the point $x_0 = (11, 11.4)$ for which the solution of (19) converges for $\ell = 2$ and does not converge for $\ell = 1$. This clearly indicates that the improvement does not only enlarge the approximation of the ROA $\Pi(Q_1^{opt}, 0)$, but also the real ROA.

Remark 3 ($\ell = 1$ vs $\ell = 2$): The previous analysis shows that using two layers in the continuous NN, yields a larger ROA compared to the single-layer case. In addition, the

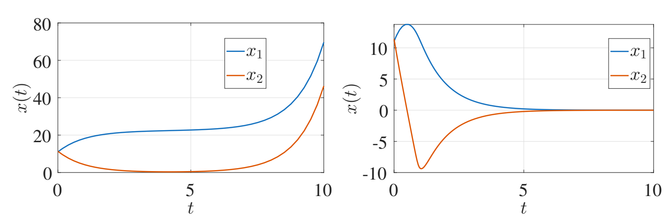

Fig. 5. Solutions of (19) with $x_0 = (11, 11.4)$, left $\ell = 1$, right: $\ell = 2$.

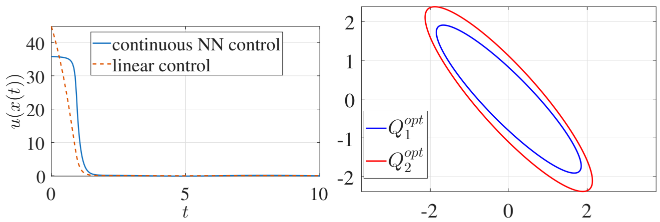

Fig. 6. Left: $u(x(t))$ right: Ellipsoids $x^\top Q_1^{opt} x = 1$, $x^\top Q_2^{opt} x = 1$.

initial transient overshoot is significantly reduced when increasing the number of layers (see Figure 2). ◦

Remark 4 (cont. NN vs linear feedback): In view of Figure 3, the initial transient overshoot obtained with the continuous NN is significantly smaller than the one produced by linear feedback control. Moreover, as illustrated in Figure 4, when the uncertainty $d(t) = (0.01, 0.1)^\top \sin(t)$ is added to the model (19), the linear feedback appears to be more sensitive than the nonlinear feedback induced by the continuous NN. This indicates that the closed-loop system with the continuous NN controller exhibits greater robustness with respect to uncertainties than the linear feedback law. ◦

V. PROOFS

A. Proof of Theorem 1

The proof of the main result is based on Lyapunov arguments. For the sake of readability, certain technical computations are formulated as claims, with their proofs postponed to Sections V-B, V-C, and V-D. We consider the following Lyapunov function

$$V(x) := (x - x_*)^\top Q (x - x_*), \quad \forall x \in \mathbb{R}^n.$$

We have by using (7) and (9)

$$\dot{V}(x,u) = \begin{bmatrix} x - x_* \\ u - u_* \end{bmatrix}^\top \begin{bmatrix} A^\top Q + QA & QB \\ B^\top Q & 0 \end{bmatrix} \begin{bmatrix} x - x_* \\ u - u_* \end{bmatrix}. \tag{25}$$

Let us now express $u - u_*$ in terms of linear and nonlinear part. For that, we define for any $i \in \{1, \dots, \ell\}$ the shifted activation function

$$\tilde{\phi}_i := \phi_i - \alpha_i I_d \tag{26}$$

and, for any $x \in \mathbb{R}^n$,

$$J_i(x) := W_{\ell+1} \circ \ldots \circ W_{i+1} \tilde{\phi}_i(v_i), \tag{27}$$

where $v_i = W_i z_{i-1}$ is given by (4), z_i by (1). The following claim holds (its proof is postponed in Section V-B).

Claim 1: *For any $x \in \mathbb{R}^n$, the control input u obtained from* (1) *is given by*

$$u(x) = \mathscr{K} x + \sum_{i=1}^{\ell} u_{i+1}^{\ell} J_i(x), \tag{28}$$

where $\mathscr{K}$ is defined in (11)*, u_i^k in* (10) *and J_i in* (27). ⋄

Using Claim 1, we have $u - u_* = \mathscr{K}(x - x_*) + \sum_{i=1}^{\ell} u_{i+1}^{\ell} (J_i(x) - J_i(x_*))$, and from (25), we get

$$\dot{V}(x) = \begin{bmatrix} \delta x \\ \delta J \end{bmatrix}^\top \begin{bmatrix} (A + B\mathscr{K})^\top Q + Q(A + B\mathscr{K}) & \star \\ M_0(Q) & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta J \end{bmatrix}$$

where

$$\delta x := x - x_* \tag{29}$$

$$\delta J := ((J_1(x) - J_1(x_*))^\top, \dots, (J_\ell(x) - J_\ell(x_*))^\top)^\top, \tag{30}$$

$$M_0(Q) := M(0, K, Q) = \left(u_2^{\ell} B^\top Q, \dots, u_{\ell+1}^{\ell} B^\top Q \right)^\top. \tag{31}$$

For brevity, we define

$$Y(x) := (\delta x, \delta J)^\top, \quad \forall x \in \mathbb{R}^n \tag{32}$$

and we have

$$\dot{V}(x) = Y^\top(x) \begin{bmatrix} (A + B\mathscr{K})^\top Q + Q(A + B\mathscr{K}) & \star \\ M_0(Q) & 0 \end{bmatrix} Y(x).$$

Adding and subtracting $M(\lambda, K, Q)$ and $T(\lambda)$ gives

$$\begin{aligned} \dot{V}(x) = & Y^\top(x) \begin{bmatrix} (A + B\mathscr{K})^\top Q + Q(A + B\mathscr{K}) & \star \\ M(\lambda, K, Q) & T(\lambda) \end{bmatrix} Y(x) \\ & - \mathscr{S}(x), \end{aligned}$$

where, for any $x \in \mathbb{R}^n$,

$$\mathscr{S}(x) := Y^\top(x) \begin{bmatrix} 0 & \star \\ M(\lambda, K, Q) - M_0(Q) & T(\lambda) \end{bmatrix} Y(x). \tag{33}$$

Thanks to the LMI (14), there exists $\varepsilon > 0$ such that

$$\dot{V}(x) \leq -\varepsilon \|Y(x)\|^2 - \mathscr{S}(x).$$

This with (29) and (32) yields, for any $x \in \mathbb{R}^n$,

$$\dot{V}(x) \leq -\varepsilon \|x - x_*\|^2 - \mathscr{S}(x). \tag{34}$$

Let us now determine the set of x for which $\mathscr{S}(x)$ is nonnegative. We first notice that the following claim, whose proof is provided in Section V-C.

Claim 2: *For any $x \in \mathbb{R}^n$, $\mathscr{S}(x)$ can be rewritten as*

$$\begin{aligned} \mathscr{S}(x) = 2 \sum_{i=1}^{\ell} \sum_{k=1}^{p} \lambda_i^k \Big(& W_{\ell+1}^k \circ \ldots \circ W_{i+1} (\delta\phi_i(v_i) - \alpha_i \delta v_i)) \\ & \times (W_{\ell+1}^k \circ \ldots \circ W_{i+1} (\beta_i \delta v_i - \delta\phi_i(v_i)) \Big), \end{aligned}$$

where $\lambda_i = (\lambda_i^1, \dots, \lambda_i^p)^\top$ and $W_{\ell+1} = (W_{\ell+1}^1, \dots, W_{\ell+1}^p)^\top$. ⋄

From Assumption 1, for any $i \in \{1, \dots, \ell\}$, ϕ_i satisfies the $W_{\ell+1} \circ \ldots \circ W_{i+1}$−sector condition $[\alpha_i, \beta_i]$ around $v_{*i} \in \mathscr{I}(\underline{v}_i, \overline{v}_i)$. Then for any $v_i \in \mathscr{I}(\underline{v}_i, \overline{v}_i)$ and $k \in \{1, \dots, p\}$ (see Definition 1)

$$\begin{aligned} & W_{\ell+1}^k \circ \ldots \circ W_{i+1} (\delta\phi_i(v_i) - \alpha_i \delta v_i)) \\ & \qquad \times W_{\ell+1}^k \circ \ldots \circ W_{i+1} (\beta_i \delta v_i - \delta\phi_i(v_i)) \geq 0. \end{aligned}$$

Combining this with Claim 2 and the fact that $\lambda_i > 0$ yields for any $v_i \in \mathscr{I}(\underline{v}_i, \overline{v}_i)$, $\mathscr{S}(x) \geq 0$. In other words, we have

$$\mathscr{S}(x) \geq 0, \quad \forall x \in \{x \in \mathbb{R}^n, \ v \in \mathscr{I}(\underline{v}, \overline{v})\}. \tag{35}$$

Using assumption (15) and the Schur complement, we have, for all $\tau \in [0,1]$ that

$$W_1L(\tau)Q^{-1}(W_1L(\tau))^\top \le (\bar{v}_1(\tau) - v_{*1}(\tau))^2.$$

This allows to apply [10, Lemma 1] to get that

$$\Pi(Q,x_*) \subseteq \{x : -\delta\bar{v}_1(\tau) \le W_1L(\tau)\delta x \le \delta\bar{v}_1(\tau), \forall \tau \in [0,1]\},$$

where $\delta v_i = v_i - v_{*i}$. Since, for all $\tau \in [0,1]$, $W_1L(\tau)\delta x = \delta v_1(\tau)$ (see (1), (2), (29)) we obtain $\Pi(Q,x_*) \subseteq \{x : -\delta\bar{v}_1 \le \delta v_1 \le \delta\bar{v}_1\}$. Using the fact that $\underline{v}_1 := 2v_{*1} - \bar{v}_1$ by assumption, we get

$$\Pi(Q,x_*) \subseteq \{x \in \mathbb{R}^n : \; \underline{v}_1 \le v_1 \le \bar{v}_1\}.$$

Combining with Remark 1, it ensures that for any $x \in \Pi(Q,x_*)$, the corresponding $v \in \mathscr{I}(\underline{v},\bar{v})$. Hence from (35), $\mathscr{S}(x) \ge 0$, for any $x \in \Pi(Q,x_*)$. Therefore, from (34)

$$\dot{V}(x) \le -\varepsilon\|x - x_*\|^2, \quad \forall x \in \Pi(Q,x_*). \tag{36}$$

And we consider the following claim stated in Section V-D.

Claim 3: *The set $\Pi(Q,x_*)$ is forward invariant, i.e., $x(0) \in \Pi(Q,x_*)$, implies that $x(t) \in \Pi(Q,x_*)$, for any $t \ge 0$.* ◇

Using this claim, we conclude that (36) holds for all $x(0) \in \Pi(Q,x_*)$. Therefore, the local exponential stability of x_* follows by Lyapunov arguments (see [13, Theorem 4.1]). Moreover, the ROA contains $\Pi(Q,x_*)$.

B. Proof of Claim 1

Let $x \in \mathbb{R}^n$. In view of (8), we have

$$u(x) = W_{\ell+1}\phi_\ell\left(W_\ell\phi_{\ell-1}(\ldots W_2\phi_1(W_1Lx)\ldots)\right).$$

Using (26), and the fact that $v_1 = W_1z_0 = W_1Lx$, it holds that

$$\phi_1(W_1Lx) = \tilde{\phi}_1(W_1Lx) + \alpha_1W_1Lx$$

and noticing that $v_2 = W_2z_1 = W_2\phi_1(v_1)$, we have

$$\begin{aligned}\phi_2(W_2\phi_1(W_1Lx)) &= \tilde{\phi}_2(W_2\phi_1(W_1Lx)) + \alpha_2W_2\phi_1(W_1Lx)\\ &= \tilde{\phi}_2(v_2) + \alpha_2W_2\tilde{\phi}_1(v_1) + \alpha_1\alpha_2W_2\circ W_1Lx.\end{aligned}$$

Similarly, since $v_3 = W_3z_2 = W_3\phi_2(W_2\phi_1(W_1Lx))$, we get

$$\begin{aligned}\phi_3(W_3\phi_2(W_2\phi_1(W_1Lx))) &= \tilde{\phi}_3(v_3) + \alpha_3W_3\tilde{\phi}_2(v_2)\\ &\quad + \alpha_2\alpha_3W_3\circ W_2\tilde{\phi}_1(v_1) + \alpha_1\alpha_2\alpha_3W_3\circ W_2\circ W_1Lx.\end{aligned}$$

Thus, by iteration we observe (since $v_i = W_iz_{i-1}$),

$$\begin{aligned}\phi_\ell\left(W_\ell\phi_{\ell-1}(\ldots W_2\phi_1(W_1Lx)\ldots)\right) &= \left(\prod_{i=1}^{\ell}\alpha_i\right)W_\ell\circ\ldots\circ W_1Lx\\ &\quad + \sum_{i=1}^{\ell-1}\left(\prod_{j=i+1}^{\ell}\alpha_j\right)W_\ell\circ\ldots\circ W_{i+1}\tilde{\phi}_i(v_i) + \tilde{\phi}_\ell(v_\ell).\end{aligned}$$

This yields,

$$\begin{aligned}W_{\ell+1}\phi_\ell\left(W_\ell\phi_{\ell-1}(\ldots W_2\phi_1(W_1Lx)\ldots)\right) &= \left(\prod_{i=1}^{\ell}\alpha_i\right)W_{\ell+1}\circ\\ \ldots\circ W_1Lx &+ \sum_{i=1}^{\ell}\left(\prod_{j=i+1}^{\ell}\alpha_j\right)W_{\ell+1}\circ\ldots\circ W_{i+1}\tilde{\phi}_i(v_i),\end{aligned}$$

where we make the convention $\prod_{j=i+1}^{\ell}\alpha_j = 1$ if $i+1 > \ell$. The conclusion follows using the notation (10), (11) and (27).

C. Proof of Claim 2

In view of (32) and (33) we have

$$\mathscr{S}(x) = 2\delta J^\top(M(\lambda,K,Q) - M_0(Q))\delta x + \delta J^\top T(\lambda)\delta J.$$

Using (13) and (30), it holds that

$$\begin{aligned}\delta J^\top T(\lambda)\delta J = \sum_{i=1}^{\ell}\Bigg(&-2\delta J_i^\top(x)\Lambda_i\delta J_i(x)\\ &+ 2\delta J_i^\top(x)\sum_{j=i+1}^{\ell}\theta_ju_{i+1}^{j-1}\Lambda_j\delta J_j(x)\Bigg).\end{aligned} \tag{37}$$

Moreover using (12) and (31), we get that

$$M(\lambda,K,Q) - M_0(Q) = \left[\theta_1u_1^0\Lambda_1K, \ldots, \theta_\ell u_1^{\ell-1}\Lambda_\ell K\right]^\top.$$

Then, replacing δJ by (30) ensures that

$$\delta J^\top(M(\lambda,K,Q) - M_0(Q))\delta x = \sum_{i=1}^{\ell}\theta_iu_1^{i-1}\delta J_i^\top(x)\Lambda_iK\delta x.$$

Combining this with (37) yields

$$\begin{aligned}\mathscr{S}(x) = 2\sum_{i=1}^{\ell}\Bigg[&\delta J_i^\top(x)\Lambda_i\left(-\delta J_i(x) + \theta_iu_1^{i-1}K\delta x\right)\\ &+ \delta J_i^\top(x)\sum_{j=i+1}^{\ell}\theta_ju_{i+1}^{j-1}\Lambda_j\delta J_j(x)\Bigg].\end{aligned}$$

We make the following observation. For $\ell = 1$, we have

$$\mathscr{S}(x) = 2\delta J_1^\top(x)\Lambda_1(-\delta J_1(x) + \theta_1u_1^0K\delta x).$$

For $\ell = 2$, we have

$$\begin{aligned}\mathscr{S}(x) &= 2\Big[\delta J_1^\top(x)\Lambda_1(-\delta J_1(x) + \theta_1u_1^0K\delta x) + \theta_2u_2^1\delta J_1^\top(x)\\ &\times\Lambda_2\delta J_2(x)\Big] + 2\delta J_2^\top(x)\Lambda_2(-\delta J_2(x) + \theta_2u_1^1K\delta x)\\ &= 2\left[\delta J_1^\top(x)\Lambda_1(-\delta J_1(x) + \theta_1u_1^0K\delta x)\right]\\ &+ 2\left[\delta J_2^\top(x)\Lambda_2\left(-\delta J_2(x) + \theta_2\left(u_1^1K\delta x + u_2^1\delta J_1(x)\right)\right)\right].\end{aligned}$$

Thus by iteration we can observe that $\mathscr{S}(x)$ reduces to

$$\begin{aligned}\mathscr{S}(x) = 2\sum_{i=1}^{\ell}\delta J_i^\top(x)\Lambda_i\Bigg(&-\delta J_i(x) + \theta_i\Bigg(u_1^{i-1}K\delta x\\ &+ \sum_{j=1}^{i-1}u_{j+1}^{i-1}\delta J_j(x)\Bigg)\Bigg).\end{aligned} \tag{38}$$

Now let us notice that

$$\beta_iv_i - \phi_i(v_i) = (\beta_i - \alpha_i)v_i - (\phi_i(v_i) - \alpha_iv_i).$$

And using (10), and (26), it holds that $\beta_iv_i - \phi_i(v_i) = \theta_iv_i - \tilde{\phi}_i(v_i)$, which thanks to (27), gives

$$W_{\ell+1}\circ\ldots\circ W_{i+1}(\beta_iv_i - \phi_i(v_i)) = \theta_iW_{\ell+1}\circ\ldots\circ W_{i+1}v_i - J_i(x).$$

Proceeding as in the proof of Claim 1, we get by induction

$$v_i = u_1^{i-1}W_i\circ\ldots\circ W_1Lx + \sum_{j=1}^{i-1}u_{j+1}^{i-1}W_i\circ\ldots\circ W_{j+1}\tilde{\phi}_j(v_j).$$

Thus using notation (11) and (27), we have

$$W_{\ell+1}\circ\ldots\circ W_{i+1}(\beta_i v_i-\phi_i(v_i)) = -J_i(x) \\ +\theta_i\left(u_1^{i-1}Kx+\sum_{j=1}^{i-1}u_{j+1}^{i-1}J_j(x)\right).$$

Thus by virtue of (27), (38) can be rewritten as

$$\mathscr{S}(x)=\sum_{i=1}^{\ell}\left(W_{\ell+1}\circ\ldots\circ W_{i+1}(\delta\phi_i(v_i)-\alpha_i\delta v_i)\right)^{\top}\Lambda_i \\ \times\left(W_{\ell+1}\circ\ldots\circ W_{i+1}(\beta_i\delta v_i-\delta\phi_i(v_i))\right).$$

Since $\lambda_i=(\lambda_i^1,\ldots,\lambda_i^p)^{\top}$ and $W_{\ell+1}=(W_{\ell+1}^1,\ldots,W_{\ell+1}^p)^{\top}$,

$$\mathscr{S}(x)=\sum_{i=1}^{\ell}\sum_{k=1}^{p}\lambda_i^k\Big(W_{\ell+1}^k\circ\ldots\circ W_{i+1}(\delta\phi_i(v_i)-\alpha_i\delta v_i) \\ \times W_{\ell+1}^k\circ\ldots\circ W_{i+1}(\beta_i\delta v_i\delta\phi_i(v_i))\Big).$$

D. Proof of Claim 3

Let $x(0)\in\Pi(Q,x_*)$, and assume by contradiction that there exists $t_0>0$ such that $x(t_0)\notin\Pi(Q,x_*)$, i.e., $V(x(t_0))>1$. Then we may define $t_*\in(0,+\infty)$ such that

$$t_*:=\inf\{t>0,\ V(x(t))>1\}. \tag{39}$$

Since the function $t\mapsto V(x(t))$ is continuous, we have that $V(x(t_*))=1$ and $V(x(t))\leq 1$, for all $t\in[0,t_*]$. Then, still using the continuity of $t\mapsto V(x(t))$ and the fact that $V(x(t_*))=1$, there exists $\eta>0$ such that $V(x(s))\leq 1$, for all $s\in[t_*,t_*+\eta)$. Hence, for any $s\in[t_*,t_*+\eta)$, $x(s)\in\Pi(Q,x_*)$ and from (36)

$$\dot V(x(s))\leq -\varepsilon\|x(s)-x_*\|^2\leq 0. \tag{40}$$

Moreover, using the characterization of t_* given by (39), it holds that there exists $t_\eta\in(t_*,t_*+\eta)$ such that

$$V(x(t_\eta))>1. \tag{41}$$

We use (40) with the fact that $t_\eta\in(t_*,t_*+\eta)$ and we get $V(x(t_\eta))-V(x(t_*))\leq\int_{t_*}^{t_\eta}\dot V(x(s))ds\leq 0$. Since $V(x(t_*))=1$, this ensures that $V(x(t_\eta))\leq 1$, which contradicts (41).

VI. CONCLUSIONS AND FUTURE WORKS

In this paper, we have established local stabilization of linear finite-dimensional systems by designing feedback controllers using neural networks with infinite number of neurons. The proposed condition incorporates a sector condition, which allows to handle the nonlinearity of the activation function. Furthermore, the method provides a systematic way to approximate the region of attraction around the equilibrium, which increases with respect to the depth of the continuous NN. We also show that the controller constructed from the continuous NN is more robust with respect to uncertainties and reduces the initial transient overshoot in comparison with a linear controller. This work extends the stability results presented in [20], underscoring the insight gained from considering infinitely many neurons. Future research directions include extending the analysis to the stability of linear partial differential equations and nonlinear systems [16].

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