STABILITY ANALYSIS OF INFINITE DIMENSIONAL SYSTEMS: THEORY AND APPLICATIONS

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Outline

1. Time-delay systems: Stability and robustness
2. Partial differential equations: Stabilization
3. The chemostat models: Applications
4. Discussions: Perspectives

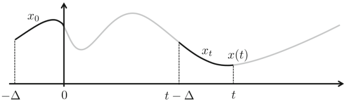
PART I

Time-delay systems: Stability and robustness

Time-delay system (TDS)

A TDS is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \tag{1}$$



$$\begin{aligned} x_t : [-\Delta, 0] &\longrightarrow \mathbb{R}^n \\ s &\longmapsto x(t+s) \end{aligned}$$

State: $x_t \in C([-\Delta, 0], \mathbb{R}^n) =: \mathcal{X}^n$

Input: $u \in L^\infty_{loc}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) =: \mathcal{U}^m$

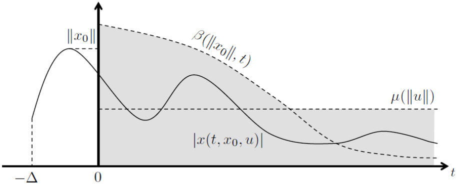
f is Lipschitz on bounded sets and $f(0,0) = 0$.

1. x_t is a function with the norm $\|x_t\| := \sup\limits_{s \in [-\Delta, 0]} |x_t(s)|$.
2. $x(t)$ is a vector in $\mathbb{R}^n$.
3. Let's state robustness stability results which resemble to finite dimension ones.

Input-to-state stability (ISS)

(1) is **ISS** if $\exists\, \beta \in \mathcal{KL}$, $\mu \in \mathcal{K}_\infty$ s.t., for all $x_0 \in \mathcal{X}^n$, $u \in \mathcal{U}^m$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$



$\beta(s,t) = kse^{-\lambda t}$ $\quad\Rightarrow\quad$ (1) is **exponentially ISS** (exp-ISS).

Figure: Class $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

ISS properties

ISS guarantees:

1. $\dot{x}(t) = f(x_t, 0)$ is globally asymptotically stable (GAS),
2. converging input implies converging state

$$\lim_{t\to+\infty} |u(t)| = 0 \quad \Rightarrow \quad \lim_{t\to+\infty} |x(t, x_0, u)| = 0,$$

3. bounded input implies bounded state

$$||u|| < +\infty \quad \Rightarrow \quad \sup_{t\geq 0} |x(t, x_0, u)| < +\infty,$$

4. asymptotic gain property

$$\limsup_{t\to+\infty} |x(t, x_0, u)| \leq \mu(||u||).$$

Lyapunov-Krasovskii-Functional (LKF)

Definition 1

$V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is a **LKF** if it is **Lipschitz on bounded sets** and $\exists\ \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ s.t.

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|), \quad \forall t \geq 0. \tag{3}$$

Coercive LKF if

$$\underline{\alpha}(\|x_t\|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|), \quad \forall t \geq 0.$$

ISS with LKF-wise dissipation

1. LKF-wise dissipation if $\exists\ \alpha, \gamma \in \mathcal{K}_\infty$ s.t

$$D^+ V \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \tag{4}$$

2. point-wise dissipation if $\exists\ \alpha, \gamma \in \mathcal{K}_\infty$ s.t.

$$D^+ V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{5}$$

$D^+ V$ is the Driver derivative of V.

Theorem 2 (Karafyllis, Pepe, and Jiang 2008)

TDS (1) is ISS if and only if it admits an ISS LKF with LKF-wise dissipation.

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ISS with point-wise dissipation?

Conjecture 1 (Chaillet, Pepe, Mason, and Chitour 2017)

Assume that the system (1) *admits a LKF with a point-wise dissipation. Then it is ISS.*

Tentative answers: Chaillet, Pepe, Mason, and Chitour (2017), Chaillet, Karafyllis, Pepe, and Wang (2023), Mironchenko, Wirth, Chaillet, and Brivadis (2024).

Motivation....

Theorem 3 (Hale and Lunel 2013, Krasovskii 1963)

If there exists a LKF V and $\alpha \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(|x(t)|), \quad \forall t \geq 0,$$

*then the system $\dot{x}(t) = f(x_t, 0)$ is **globally asymptotically stable (GAS)**.*

ISS with point-wise dissipation?

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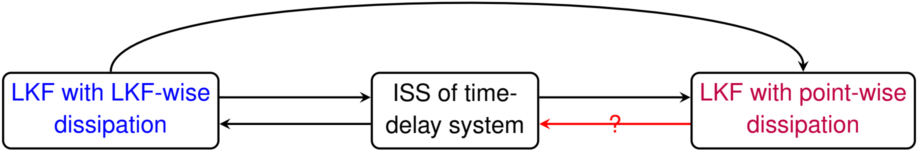
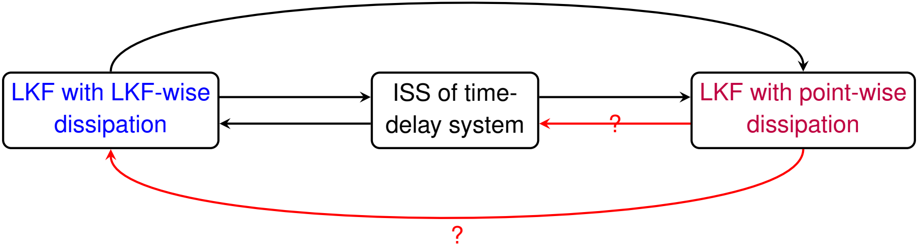


LKF with LKF-wise dissipation

ISS of time-delay system

LKF with point-wise dissipation

ISS with point-wise dissipation?



LKF with LKF-wise dissipation

ISS of time-delay system

LKF with point-wise dissipation

A trick to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} V_2(x_t(s))ds, \tag{6}$$

which dissipates point-wisely, i.e

$$D^+ V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding ke^{cs} in the integral part of (57), we get

$$W(x_t) := V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

Proposition 1 (Loko, Chaillet, Wang, Karafyllis, and Pepe (2025))

If $\alpha(|x|) \geq pV_2(x)$, $(p > 0)$ then W is an ISS LKF with LKF-wise dissipation.

Corollary: $(V_1, V_2, \alpha$ quadratics)=Orłowski, Chaillet, Destexhe, and Sigalotti (2022).

A Counter-example

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{7}$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^{0} x_t(s)^4 ds,$$
$$W(x_t) := \frac{1}{4}x(t)^4 + \int_{-\Delta}^{0} ke^{cs} x_t(s)^4 ds.$$

Proposition 2 (Loko, Chaillet, Wang, Karafyllis, and Pepe 2025)

1. *System* (7) ***is ISS****.*
2. *V is an ISS LKF with* ***point-wise dissipation*** *for* (7)*.*
3. *Given any $k, c > 0$, W is not a LKF with LKF-wise dissipation.*

ISS under point-wise dissipation

Theorem 4 (Loko, Chaillet, Wang, Karafyllis, and Pepe 2025)

Assume that there exists a LKF V which dissipates point-wisely as

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \tag{8}$$

Assume that $\exists\, \sigma \in \mathcal{K}_\infty$, s.t.

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{9}$$

Then, if

$$\liminf_{r\to+\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{10}$$

the system (1) *is ISS.*

1. Given Q, (9) is always satisfied.
2. α or σ polynomial $\Rightarrow$ no delay in (10).
3. Theorem 4 extends the existing ones.

Some difficulties with LKF

1. LKF-wise dissipation may appear hard to get

2. V is a functional, not a function $\Rightarrow$ deal with infinite dimensional tools

Let's see a Lyapunov approach which relies on functions and not on LKFs.

ISS from Razumikhin

Theorem 5 (Teel 1998)

Let $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ *positive definite and radially unbounded. If*

$$V_0(x(t)) \geq \max\left\{\rho\left(\max_{s\in[-\Delta,0]} V_0(x_t(s))\right), \gamma(|u(t)|)\right\} \Rightarrow \dot{V}_0(x(t)) \leq -\alpha(|x(t)|), \quad (11)$$

with $\rho(s) < s$ *for all* $s > 0$, *then the TDS* (1) *is ISS.*

1. Like a small gain result: all the delays term are treated as perturbations.
2. Just a sufficient condition, so less general than Theorem 2.
3. Theorem 5 is based on function, so resembles to finite dimension setting.

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Question and motivations

How to link both Razumikhin and LKF approaches?

Some motivating reasons:

1. Exploit LKF advantages under Razumikhin assumptions.
2. Alternative proof of LKF converse result, combining both Razumikhin and LKF.

Coercive LKF from Razumikhin

Theorem 6 (Loko, Chaillet, and Karafyllis 2024)

Let $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ *satisfying* (11) *with*

$$\rho_0 := \sup_{s>0} \frac{\rho(s)}{s} < 1.$$

Then the functional

$$V(x_t) := \max_{s\in[-\Delta,0]} \rho_0^{-\tau/\Delta} V_0(x_t(s)) \tag{12}$$

is a coercive ISS LKF of (1).

1. Combine Razumikhin and LKF
2. Getting explicit and coercive LKF
3. Advantage of coercivity: interchanging V and $\|x_t\|$.

Coercive LKF from Razumikhin

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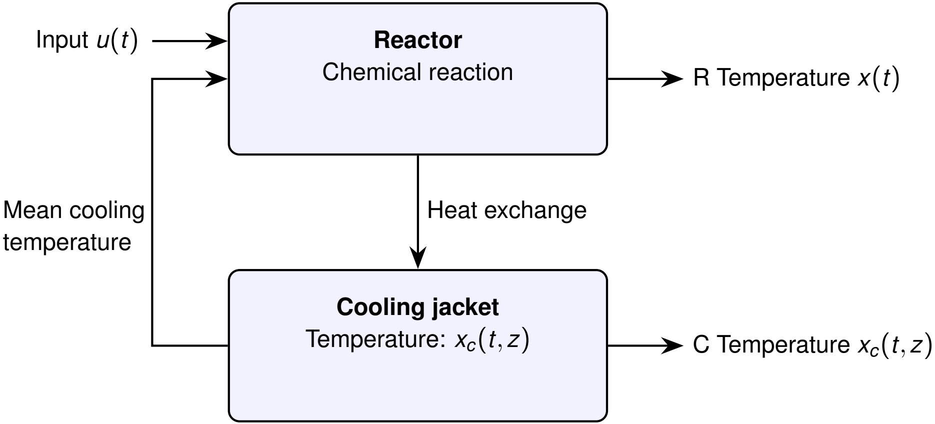
Application: Chemical reactor model

Figure: Chemical reactor with cooling device (generated by Mistral AI)

Application: Chemical reactor model



Input $u(t)$

Reactor

Chemical reaction

R Temperature $x(t)$

Heat exchange

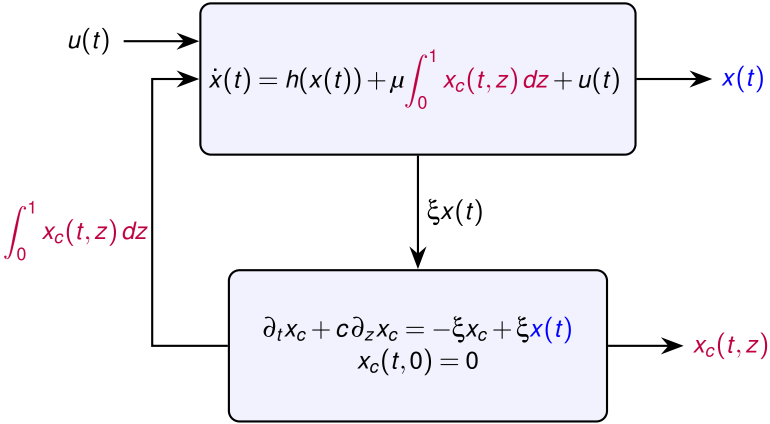
Mean cooling temperature

Cooling jacket

Temperature: $x_c(t,z)$

C Temperature $x_c(t,z)$

Application: Chemical reactor model



$u(t)$

$$\dot{x}(t) = h(x(t)) + \mu \int_0^1 x_c(t,z)\,dz + u(t)$$

$x(t)$

$\xi x(t)$

$\int_0^1 x_c(t,z)\,dz$

$$\partial_t x_c + c\partial_z x_c = -\xi x_c + \xi x(t)$$
$$x_c(t,0) = 0$$

$x_c(t,z)$

μ, c, ξ are positive constants.

$h(x) = g(x) - (\mu+1)x$.

PDE-ODE into PDE-TDS

$$S: \; w \mapsto x_c(z) = Sw(z) := \xi \int_0^{z/c} e^{-\xi s} w(cs)\,ds.$$

$$\begin{cases} \partial_t x_c + c\partial_z x_c = -\xi x_c + \xi x(t) \\ x_c(t,0) = 0, \\ \dot{x} = h(x) + \mu \int_0^1 x_c(t,z)dz + u \end{cases} \Rightarrow \begin{cases} \partial_t w + c\partial_z w = 0 \\ w(t,0) = x(t) \\ \dot{x} = h(x) + \mu\xi \int_0^1 \int_0^{\Delta z} e^{-\xi s} x(t-s)\,ds\,dz + u. \end{cases}$$

$$\Delta := 1/c.$$

Exp-ISS of chemical reactor model

$$\begin{cases} \partial_t x_c(t,z) + c\partial_z x_c(t,z) = -\xi x_c(t,z) + \xi x(t) \\ \dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu \int_0^1 x_c(t,z)dz + u(t) \\ x_c(t,0) = 0, \quad \forall t \geq 0. \end{cases} \tag{13}$$

Proposition 3 (Loko, Chaillet, and Karafyllis 2024)

Assume that

$$\sup_{x \neq 0} \frac{|g(x)|}{|x|} < 1 + \frac{\mu c}{\xi}\left(1 - e^{-\xi/c}\right). \tag{14}$$

Then the system (13) *is exp-ISS.*

1. Proposition 3 extends the GES in **Karafyllis and Krstic (2021)**.
2. The analysis used sup norm, not L^p norm.

Concluding remarks

1. Exponential trick does not work systematically.
2. Point-wise dissipation ensures ISS under growth condition.
3. The conjecture is still an open question.
4. Coercive LKF can be deduced also from Halanay approach: Loko, Chaillet, and Karafyllis (2024).
5. An application is given for a chemical reactor model.

PART II

Partial differential equations: Stabilization

General PDE system

Let $(X, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. We consider

$$\partial_t y = \mathcal{A} y + B w, \tag{15}$$

where

- $\mathcal{A}$ and B are linear operators;
- $y(t, \cdot) \in X$ the state of the system;
- $w(t) \in \mathbb{C}$ the control input.

Feedback stabilization

Conditions on $\mathcal{A}$ and B to exhibit $\mathcal{K} \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$ s.t $w(t) = \mathcal{K}y(t,\cdot)$, and:

1. the closed-loop system
$$\partial_t y = \mathcal{A}y + B\mathcal{K}y, \tag{16}$$
is well-posed in X;

2. any solution of (16) satisfies
$$\|y(t,\cdot)\|_X \leq ke^{-\mu t}\|y(0,\cdot)\|_X, \quad \forall t \geq 0.$$

Existing methods: Pole shifting, Control Lyapunov functions, Optimal control method,...

F-equivalence method

Assume that $\mathcal{A}$ generates a C_0 **semi-group**.Then, $\exists\ \omega \in \mathbb{R}$, $k \geq 1$ s.t. for any $\lambda > 0$ the system

$$\partial_t v = (\mathcal{A} - \lambda I)v, \tag{17}$$

is well-defined and

$$\|v(t,\cdot)\|_X \leq k e^{-(\lambda-\omega)t} \|v(0,\cdot)\|_X, \quad \forall t \geq 0. \tag{18}$$

If $\exists$ an isomorphism T and feedback $\mathcal{K}$ s.t

$$\partial_t y = (\mathcal{A} + B\mathcal{K})y \quad \underset{v=Ty}{\Longleftrightarrow} \quad \partial_t v = (\mathcal{A} - \lambda I)v$$

then

$$\|y(t,\cdot)\|_X \leq K e^{-(\lambda-\omega)t} \|y(0,\cdot)\|_X, \quad \forall t \geq 0. \tag{19}$$

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Transformations T

1. When T is a Volterra transformation of the second kind: **Balogh and Krstic (2002)**, **Bošković, Balogh, and Krstić (2003)**, **Krstic and Smyshlyaev (2008)**.

2. When T is a Fredholm transformation. **Coron and Lü (2014)**, **Coron and Lü (2015)**.

The conjecture

Conjecture 2

Assume that $(\mathcal{A}, B)$ *is* ***exactly controllable and admissible*** *in a certain Hilbert space* H. *Then, for any* $\lambda > 0$ $\exists$ *a unique pair* $(T, \mathcal{K})$ *such that* $TB = B$ *and* T *transforms*

$$\partial_t y = \mathcal{A} y + B \mathcal{K} y,$$

into

$$\partial_t v = \mathcal{A} v - \lambda v.$$

The conjecture is solved in finite dimension: Brunovsky (1970), Coron (2015).

1. **Admissibility:** For any control, the system is well-posed.
2. **Exact Controllability:** There exists a control to start from any y_0 to reach any y_T.

Answer in infinite dimension?

1. **The linear Schrödinger equation:** Coron, Gagnon, and Morancey (2018)
2. **Degenerate parabolic:** Gagnon, Lissy, and Marx (2021) and Lissy and Moreno (2023)
3. **The transport equation:** Zhang (2022)
4. **The linearized Saint-Venant system:** Coron, Hayat, Xiang, and Zhang (2022)
5. **The 1D heat equation** Gagnon, Hayat, Xiang, and Zhang (2022)
6. **The skew-adjoint systems:** Gagnon, Hayat, Xiang, and Zhang (2025)

These results benefit at least one of these facts:

- the involved operator $\mathcal{A}$ is self or skew-adjoint,
- one can explicitly express the eigenvectors of $\mathcal{A}$.

Spectral assumptions on $\mathcal{A}$

Can we extend these results to any general "**spectral**" operator $\mathcal{A}$?

1. $\mathcal{A}$ generates a dissipative C^0 semi-group on X.
2. The eigenvectors φ_n of $\mathcal{A}$ form a Riesz basis of X.
3. The eigenvalues λ_n have **finite multiplicity** and there exists $\alpha > 1$ s.t.

$$cn^{\alpha} \leq |\lambda_n| + 1 \leq Cn^{\alpha}, \quad \forall n \in \mathbb{N}^*, \tag{20}$$

$$|\lambda_n - \lambda_p| \geq C_1 n^{\alpha-1} |n-p|, \quad \forall n, p \in \mathbb{N}^*. \tag{21}$$

$\alpha = 1$ is still an open problem.

Let $(\widetilde{\varphi}_n)_n$, the bi-orthogonal family of $(\varphi_n)_n$ as $\langle \varphi_n, \widetilde{\varphi}_m \rangle = \delta_{nm}$.

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Main result

Theorem 7 (Hayat and Loko 2024)

Let $\gamma \in [0, (\alpha - 1)/2)$. *If*

$$c_1 \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{\gamma}, \quad \forall n \in \mathbb{N}^*. \tag{22}$$

Then, for any $\lambda > 0$, $\exists$ $\mathcal{K}$ *and* T *such that* $TB = B$ *and* T *maps*

$$\partial_t y = \mathcal{A} y + B \mathcal{K} y, \tag{23}$$

to

$$\partial_t v = \mathcal{A} v - \lambda v.$$

And (23) *is exponentially stable in* H^r, $r \in \left(\frac{1}{2} - \alpha + \gamma, \alpha - \frac{1}{2} - \gamma\right)$.

1. (22) with $\gamma = 0$ used in **Gagnon, Hayat, Xiang, and Zhang (2022)**, **Gagnon, Hayat, Xiang, and Zhang (2025)**.
2. (22) is less conservative requirement as γ is not necessarily 0.
3. $\mathcal{A}$ is not assumed to be self nor skew-adjoint.

Main result

Theorem 8 (Hayat and Loko (2024))

Let $\gamma \in \left[0, \frac{\alpha-1}{2}\right)$, $r \in \left(\frac{1}{2} - \alpha + \gamma, \alpha - \frac{1}{2} - \gamma\right)$. *If*

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^* \tag{24}$$

then $\exists\ \mathcal{K}$ *s.t. the system*

$$\partial_t y = \mathcal{A} y + B \mathcal{K} y$$

is exponentially stable in the study space X.

Relaxed controllability and admissibility

Assume that $\mathcal{A}$ is **skew-adjoint**.

Lemma 1 (Weiss and Xu 2011, Russell and Weiss 1994)

If B is admissible and if the system is in addition exactly controllable in X, then

$$c \leq |\langle B, \widetilde{\varphi}_n\rangle| \leq C, \ \forall\, n \in \mathbb{N}^*. \tag{25}$$

1. Admissibility and exact controllability in X imply (24) for $r = \gamma = 0$.

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n\rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^*.$$

2. What if there is no exact controllability in X, or if B is not admissible?

Lemma 2 (Hayat and Loko 2024)

Assume that (24) holds, i.e

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n\rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^*.$$

- *If* $r \in \left(\frac{1}{2} - \alpha + \gamma, -\gamma\right)$, *then the system is not exactly controllable in X.*
- *If* $r \in \left(0, \alpha - \frac{1}{2} - \gamma\right)$, *then B is not admissible with respect to X.*

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$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^*.$$

- *If $r \in \left(\frac{1}{2} - \alpha + \gamma, -\gamma\right)$, then the system is not exactly controllable in X.*
- *If $r \in \left(0, \alpha - \frac{1}{2} - \gamma\right)$, then B is not admissible with respect to X.*

Relaxed controllability and admissibility

Assume that $\mathcal{A}$ is **skew-adjoint**.

Lemma 1 (Weiss and Xu 2011, Russell and Weiss 1994)

If B is admissible and if the system is in addition exactly controllable in X, then

$$c \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq C, \ \forall\, n \in \mathbb{N}^*. \tag{25}$$

1. Admissibility and exact controllability in X imply (24) for $r = \gamma = 0$.

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^*.$$

2. What if there is no exact controllability in X, or if B is not admissible?

Lemma 2 (Hayat and Loko 2024)

Assume that (24) *holds, i.e*

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^*.$$

- *If* $r \in \left(\frac{1}{2} - \alpha + \gamma, -\gamma\right)$, *then the system is not exactly controllable in* X.
- *If* $r \in \left(0, \alpha - \frac{1}{2} - \gamma\right)$, *then* B *is not admissible with respect to* X.

Stabilization without exact controllability in X

What if there is no exact controllability in X?

$\mathcal{A}$ is still assumed to be skew adjoint.

1. From Trélat, Wang, and Xu (2019) or Zabczyk (2020), stabilization is not possible with $\mathcal{K} \in \mathcal{L}(X, \mathbb{C})$.
2. From Theorem 8, stabilization is possible with feedback $\mathcal{K} \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$.

Liu, Wang, Xu, and Yu (2022), Ma, Wang, and Yu (2023) used $\mathcal{K} \in \mathcal{L}(D(\mathcal{A}), \mathbb{C})$.

$D(\mathcal{A}) = H^{\alpha}$.

Concluding remarks

1. No admissibility of B with respect to X may lead to stabilization.
2. Exact controllability and admissibility not necessarily in X are enough to get stabilization.
3. Application to: diffusion system, Schrödinger equation, nonlinear Burger's equation,...

PART III

The chemostat model: Application

The chemostat: A microbial bioreactor

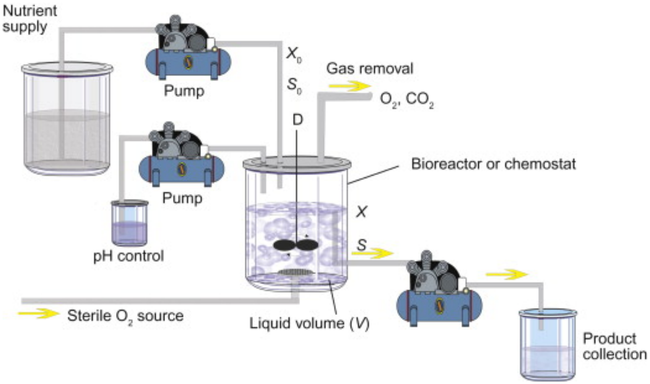


Figure: Chemostat model (borrowed from Maier and Pepper (2015)).

1. to produce antibiotics, proteins, biomass
2. to treat water (degradation of pollutant),...

A simplest chemostat

The simplest chemostat is modeled by:

$$\begin{cases} \dot{X} = (p_0\mu(S) - b - D)X \\ \dot{S} = D(S_{\text{in}} - S) - \mu(S)X. \end{cases} \tag{26}$$

Symbols	Descriptions
X	Microbial concentration or biomass
S	Concentration of the substrate (nutrient)
S_{in}	Inlet concentration of the nutrient
μ	Specific growth rate
D	Dilution rate
b	Mortality rate
p_0	Yield factor

1. Presence of mortality can alter favorable operating conditions and add complexity.
2. X, S, D, b, p_0 are positive and $\mu \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ with $\mu(0) = 0$.

Objective:

Maintain the system in a non washout steady state.

Washout point $=$ trivial equilibrium $(0, S_{\text{in}})$.

Equilibrium point

An equilibrium point of the chemostat model (26) is $(X^*, S^*) \in [0, +\infty) \times (0, S_{\text{in}}]$ s.t.

$$\begin{cases} \mu(S^*) = \frac{b+D^*}{p_0}, \\ X^* = \frac{D^*(S_{\text{in}}-S^*)}{\mu(S^*)}. \end{cases} \tag{27}$$

Given $D^* > 0$,

1. there may exist **many equilibrium points**,
2. Using the linearization of (26), an equilibrium (X^*, S^*) is **unstable** if $\mu'(S^*) < 0$.

The Monod case

When considering **Monod kinetics**:

$$\mu(S) := \frac{\mu_{\max} S}{K + S}. \tag{28}$$

Theorem 9 (Dali-Youcef, Rapaport, and Sari 2022)

There exists a non washout equilibrium point of (26)-(28) *which is GAS and locally exponentially stable.*

See also Mailleret and Bernard (2001), Karafyllis, Kravaris, Syrou, and Lyberatos (2008) for the case $b = 0$.

The Haldane case

When, considering the **Haldane kinetics**:

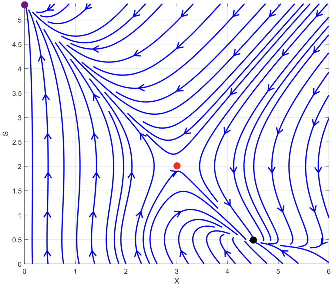
$$\mu(S) := \frac{\mu_{\max} S}{K + S + aS^2}. \tag{29}$$

Theorem 10

The system (26)-(29) *admits two non washout equilibrium points* (X^*, S^*), (X^{**}, S^{**}) *such that*

- (X^*, S^*) *is unstable,*
- (X^{**}, S^{**}) *is locally exponentially stable.*

Unstable open-loop chemostat model



Locally exponentially stable equilibrium. point: $(4.5, 0.5)$

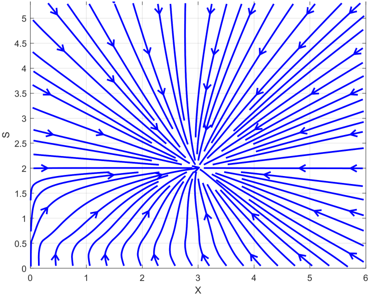
Unstable equilibrium point: $(3, 2)$

Washout point: $(0, 5.5)$

Stable closed-loop model

$$D(X,S) = \frac{D^*\mu(S)X}{\mu(S^*)X^*} + \frac{\delta b}{(\mu(S^*))^{1+\alpha}} \begin{cases} |\mu(S)-\mu(S^*)|^{1+\alpha}, & \text{if} \quad S \leq S^* \\ 0, \quad \text{if} \quad S > S^* \end{cases} \tag{30}$$

1. Mailleret and Bernard (2001) already used (30) with $b = 0$.
2. Stable equilibrium point: $(3, 2)$.



Sufficient condition for GAS

Theorem 11 (Karafyllis, Loko, Krstic, and Chaillet 2025)

Assume that there exist $X^* > 0, \quad S^* > 0$ *and*

$$p_0\mu(S) > b, \quad \forall S \in [S^*, S_{in}]. \tag{31}$$

Then the feedback law (30) *achieves GAS of* (X^*, S^*).

If $\alpha > 0$, (30) *also achieves local exponential stabilization of* (X^*, S^*).

Karafyllis and Jiang (2012) already used (31) for time-delay chemostat model.

Necessary condition for GAS

Theorem 12 (Karafyllis, Loko, Krstic, and Chaillet 2025)

If $\exists\ \overline{S} \in (S^*, S_{in})$ *s.t.*

$$\begin{aligned} p_0\mu(\overline{S}) &< b \\ \mu'(S) &\leq 0, \quad \forall S \in [\overline{S}, S_{in}]. \end{aligned}$$

Then, there is no $D(X,S) \geq 0$ *that achieves global stabilization of* (X^*, S^*).

Concluding remarks

1. The results are extended to the case of an age structured chemostat model: **Karafyllis, Loko, Krstic, and Chaillet (2025)**

2. Consider higher dimension chemostat model.

3. Include time dependancy in b or/and in S_{in}.

General Conclusion

For TDS:

1. less conservative condition is proposed to get ISS under point-wise dissipation,
2. coercive LKF is constructed from Razumikhin function,
3. application to a chemical reactor

For PDE:

1. stabilization is shown with F-equivalence method,
2. relaxed spectral conditions are used,
3. "weak" controllability and admissibility assumptions are considered.

For the chemostat:

1. feedback law is constructed in the presence of non-zero mortality,
2. stabilization of the nontrivial equilibrium is obtained.
3. the obtained condition is sufficient and necessary

Perspectives for TDS part

Let us start with this question:

Conjecture 3

Assume that the TDS (1) *admits an LKF V such that for all $t \geq 0$*

$$\underline{a}|x(t)|^2 \leq V(x_t) \leq \overline{a}\|x_t\|^2,$$
$$D^+ V \leq -a|x(t)|^2.$$

Then (1) *is* ***globally exponentially stable (GES)****.*

1. From Grüne, Sontag, and Wirth (1999), GAS ⇔ GES for ODE systems with non standard transformation.
2. Is there a "non standard transformation T" such that GAS ⇔ $T-$ GES for TDS?

Perspectives for PDE part

The compactness duality method employed does not cover the case $\alpha = 1$.

The reason is that we strongly used the convergence of $\displaystyle\sum_{n\in\mathbb{N}^*} \frac{1}{|\lambda_n|+1}$, $\lambda_n \simeq n^\alpha$.

$\alpha = 1$: Zhang (2022) for the transport equation and Coron, Hayat, Xiang, and Zhang (2022) for the water tank system.

Zhang (2022), Coron, Hayat, Xiang, and Zhang (2022) strongly used the fact that the involved eigenvectors can be explicitly estimated.

We start with the extension of Coron, Hayat, Xiang, and Zhang (2022) and we already show that the eigenvalues form a generalized Riesz basis.

Thank you for your attention!

1. E. Loko, A. Chaillet, and I. Karafyllis (2024). “Building coercive Lyapunov–Krasovskii functionals based on Razumikhin and Halanay approaches”. In: *International Journal of Robust and Nonlinear Control*
2. E. Loko, A. Chaillet, Y. Wang, I. Karafyllis, and P. Pepe (2025). “Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation”. In: *63rd IEEE Conference on Decision and Control (CDC 2024)*. IEEE
3. A. Hayat and E. Loko (2024). “Rapid stabilization of general linear systems with F-equivalence”. In: *Under review*
4. I. Karafyllis, E. Loko, M. Krstic, and A. Chaillet (2025). “Global Stabilization of Chemostats with Nonzero Mortality and Substrate Dynamics”. In: *arXiv preprint arXiv:2502.09310*
5. E. Loko and A. Chaillet (2025). “Converse Lyapunov result for exponential ISS of time-delay systems”. In: *In preparation*

Consider

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \mathcal{A}(x) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{32}$$

where

$$\mathcal{A}(x) := \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & -\lambda_2(x) \end{pmatrix} \partial_x + \delta(x) \begin{pmatrix} 1 & \frac{1}{3} \\ -\frac{1}{3} & -1 \end{pmatrix}. \tag{33}$$

1. $\lambda_1(x) = \lambda_2(x) = \lambda$ is studied in Coron, Hayat, Xiang, and Zhang (2022).
2. The family of generalized eigenvectors of $\mathcal{A}(x)$ forms a Riesz basis.

Driver derivative

Given a functional $V : \mathcal{X}^n \to \mathbb{R}^n$, its Driver derivative $D^+ V : \mathcal{X}^n \times \mathbb{R}^n \to [-\infty, +\infty]$ is defined for all $(\phi, w) \in \mathcal{X}^n \times \mathbb{R}^n$ as

$$D^+ V(\phi, w) := \lim_{h \to 0^+} \frac{V(\phi_{h,w}) - V(\phi)}{h},$$

where the function $\phi_{h,w}$ is defined by

$$\phi_{h,w}(\tau) := \begin{cases} \phi(\tau + h) & \text{if } \tau \in [-\Delta, -h] \\ \phi(0) + (\tau + h) w & \text{if } \tau \in (-h, 0]. \end{cases}$$

Sketch of the proof of the counter-example

Here $\Delta = 1$, we set $\mathcal{X} := C^0([-1,0],\mathbb{R})$ and we denote by f the dynamics of (7).

$$W \text{ dissipates LKF-wise} \Leftrightarrow \; D^+W(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|), \quad \forall \phi \in \mathcal{X}, v \in \mathbb{R}.$$

In particular for $v = 0$, this implies

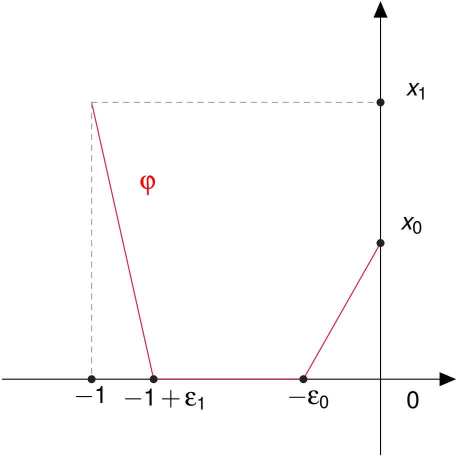
$$W \text{ dissipates LKF-wise} \Rightarrow \; D_0^+W := D^+W(\phi, f(\phi, 0)) \leq -\alpha(V(\phi)) \leq 0, \quad \forall \phi \in \mathcal{X}.$$

Sketch of the proof of the counter-example

Set $x_0 := \phi(0),\ x_1 := \phi(-1)$.

$$D_0^+ W = x_0^4 \left(k - 1 - \frac{1}{1 + x_0^2} \right) + x_1^4 \left(\frac{x_0^3}{1 + |x_0|^3} - ke^{-c} \right) - kc \int_{-1}^{0} e^{cs} \phi(s)^4 ds.$$

$D^+ W(\varphi, f(\varphi, 0)) > 0.$



Why adding exponential?

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} V_2(x_t(s))ds,$$

$$W(x_t) := V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

The Driver derivative of V and W along the solutions of (1) reads,

$$\dot{V} = \nabla V_1(x(t)) f(x_t, u(t)) + V_2(x(t)) - V_2(x_t(-\Delta)),$$

$$\dot{W} = \nabla V_1(x(t)) f(x_t, u(t)) + kV_2(x(t)) - ke^{-\Delta c} V_2(x_t(-\Delta)) - c\int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

Proof of the ISS results

Lemma 3 (Karafyllis and Jiang 2011)

Given $V_0 \in C^1(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ *and* $c > 0$*, the functional* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *defined as*

$$V(\phi) := \max_{\tau \in [-\Delta, 0]} e^{c\tau} V_0(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n,$$

is Lipschitz on bounded sets of $\mathcal{X}^n$ *and satisfies, for all* $\phi \in \mathcal{X}^n$ *and all* $v \in \mathbb{R}^m$*,*

$$\begin{aligned} V(\phi) > V_0(\phi(0)) \quad &\Rightarrow \quad D^+ V(\phi, f(\phi, v)) \leq -cV(\phi), \\ V(\phi) = V_0(\phi(0)) \quad &\Rightarrow \quad D^+ V(\phi, f(\phi, v)) \leq \max\{-cV(\phi), \nabla V_0(\phi(0)) f(\phi, v)\}. \end{aligned}$$

Sufficient conditions for admissibility and controllability

Being admissible (in X) means that there exists $\tau > 0$ and $C_\tau > 0$ such that for every $z \in \mathcal{D}(\mathcal{A}^*)$

$$\int_0^\tau |B^* S(t)^* z|^2 dt \leq C_\tau \|z\|_X^2, \tag{34}$$

where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ and $S(t)^*$ is the adjoint of $S(t)$, where $\{S(t)\}_{t \geq 0}$ is the C_0 semigroup generated by $\mathcal{A}$.

Being exactly controllable in X at time τ is equivalent to the existence of c_τ such that

$$\int_0^\tau |B^* S(t)^* z|^2 dt \geq c_\tau \|z\|_X^2. \tag{35}$$

F-equivalence method

$v = Ty$ implies

$$\begin{aligned} \partial_t v &= T \partial_t y \\ &= T(\mathcal{A} + BK) y \\ &= (\mathcal{A} - \lambda) Ty \\ &= (\mathcal{A} - \lambda) v, \end{aligned}$$

Then, we want

$$T(\mathcal{A} + BK) = (\mathcal{A} - \lambda) T, \tag{36}$$

Because the operator equality (36) has no uniqueness in the solutions we add a condition of the form

$$TB = B,$$

and the operator equality (36) becomes

$$T\mathcal{A} + BK = (\mathcal{A} - \lambda) T. \tag{37}$$

Step of the proof

$$T : \varphi_n \to -K(\varphi_n) \sum_{p\in\mathbb{N}^*} \frac{b_p\varphi_p}{\lambda_n - \lambda_p + \lambda} \tag{38}$$

where $b_p := \langle B, \widetilde{\varphi}_p\rangle$.

1. Show that the operator
$$S : \varphi_n \to \sum_{p\in\mathbb{N}^*} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda}$$
is a Fredholm operator of index 0.
2. Show that $\ker(S) = \ker(S^*) = \{0\}$ and consequently S is an isomorphism.
3. Find an explicit candidate K such that $TB = B$.
4. Show that the linear operator T is bounded and satisfies the operator equality.
5. Show that T transforms a Riesz basis into another one and then it is an isomorphism.

The feedback operator

Remark 1

The constructed feedback law is given as

$$K : n^{-\alpha/2}\varphi_n \to n^{-\alpha/2}K_n$$

where $K_n := -(\lambda + k_n)/\langle B, \widetilde{\varphi_n}\rangle$ *with* k_n *the solution of the following equation:*

$$\sum_{n\in\mathbb{N}^*} \frac{k_n}{\lambda_n - \lambda_p + \lambda} = -\sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{\lambda}{\lambda_n - \lambda_p + \lambda}\ \forall p \in \mathbb{N}^*. \tag{39}$$

Link with controllability and admissibility

The condition is

$$c_1 n^r \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \quad \forall n \in \mathbb{N}^*.$$

We have

$$|\langle B, \widetilde{\varphi}_n \rangle| \geq c_1 n^r \Leftrightarrow |\langle B, n^{-r} \widetilde{\varphi}_n \rangle| \geq c_1 .$$

We then require the controllability with respect to the space H^r since $(n^{-r}\widetilde{\varphi}_n)_n$ is a Riesz basis of H^r.
If

$$|\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma},$$

then

$$|\langle B, n^{-(r+\gamma)} \widetilde{\varphi}_n \rangle| \leq c_2 .$$

This is an admissibility property in the space $H^{r+\gamma}$.