

# ISS FOR TDS WITH POINT-WISE DISSIPATION AND F-EQUIVALENCE FOR GENERAL LINEAR PDE: APPLICATIONS

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## PART I

# Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation

Joint work with

Antoine Chaillet, Yuan Wang, Iasson Karafyllis, and Pierdomenico Pepe

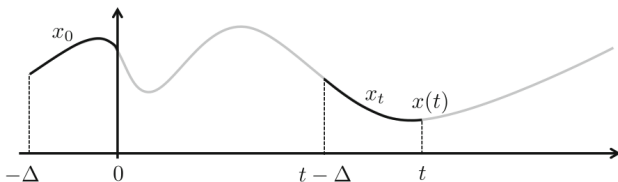
# Time-delay system

## Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \quad (1)$$

$u \in L_{loc}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ ,  $\Delta > 0$ ,  $x_t : [-\Delta, 0] \rightarrow \mathbb{R}^n$ ,  $x_t(s) = x(t+s)$ ,  $\forall s \in [-\Delta, 0]$ . Assume  $f$ , Lipschitz on bounded sets and  $f(0, 0) = 0$ , in what follows.



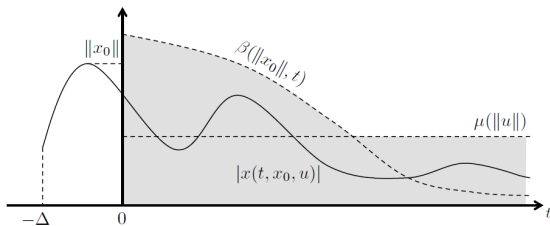
# Input-to-state stability (ISS)

## Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist  $\beta \in \mathcal{KL}$  and  $\mu \in \mathcal{K}_\infty$  such that, for all  $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$  and all  $u \in L_{loc}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ ,

$$\|x(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \mu(\|u_{[0,t]}\|), \quad \forall t \geq 0. \quad (2)$$

$$\|x(t, x_0, u)\| \leq \max\{\beta(\|x_0\|, t), \mu(\|u_{[0,t]}\|)\}, \quad \forall t \geq 0.$$



## Definition 3

A functional  $V : C([-Δ, 0], \mathbb{R}^n) \rightarrow \mathbb{R}_{\geq 0}$  is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_{\infty}$  such that, for all  $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \bar{\alpha}(\|x_t\|). \quad (3)$$

$$|x(t)| := \left( \sum_{i=1}^n |x_i(t)|^2 \right)^{1/2} \quad \text{and} \quad \|x_t\| := \sup_{\tau \in [-\Delta, 0]} |x_t(\tau)|.$$

## Definition 4

For TDS (1), an LKF  $V$  is said to be

- 1 an ISS LKF with **LKF-wise** dissipation if there exist  $\alpha, \gamma \in \mathcal{K}_\infty$  such that

$$D^+V \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad (4)$$

- 2 an ISS LKF with **point-wise** dissipation if there exist  $\alpha, \gamma \in \mathcal{K}_\infty$  such that

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \quad (5)$$

$D^+V$  is the Driver's derivative of the functional  $V$  along the solution of (1).

[Karafyllis et al., 2008]

## Theorem 5

*TDS (1) is ISS if and only if it admits an ISS LKF with **LKF-wise dissipation**.*

Is the point-wise dissipation enough to ensure ISS of TDS?

[Chaillet et al., 2017]

## Conjecture 1

*Assume that the system (1) admits a LKF with a point-wise dissipation. Then it is ISS.*

In response,

- 1 [Chaillet et al., 2017]
- 2 [Chaillet et al., 2023]

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# A way to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^0 V_2(x_t(s)) ds, \quad (6)$$

which dissipates point-wisely as follow

$$D^+ V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding  $ke^{cs}$  in the kernel of integral part of (6), we get the LKF

$$W(x_t) = V_1(x(t)) + \int_{-\Delta}^0 ke^{cs} V_2(x_t(s)) ds.$$

## Proposition 1

*If  $\alpha(|x(t)|) \geq pV_2(x(t))$ , ( $p > 0$ ) then  $W$  is a LKF with LKF-wise dissipation and the TDS (1) is ISS.*

Does this trick work systematically?

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \quad (7)$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^0 x_t(s)^4 ds,$$

$$W(x_t) := \frac{1}{4}x(t)^4 + \int_{-\Delta}^0 ke^{cs} x_t(s)^4 ds.$$

## Proposition 2

- 1 System (7) **is ISS**.
- 2  $V$  is an ISS LKF with **point-wise dissipation** for (7)
- 3 Given any  $k, c > 0$ ,  $W$  **is not a LKF with LKF-wise dissipation**.

## Theorem 6

Assume that there exists a LKF  $V$  which dissipates *point-wisely* as follow,

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \quad (8)$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \quad (9)$$

Then, if

$$\liminf_{r \rightarrow +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \quad (10)$$

the system (1) is ISS.

## Corollary 7

Assume that there exists a LKF  $V$  which dissipates *point-wisely* as follow,

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \quad (11)$$

Assume that  $Q(\cdot) := \alpha(|\cdot|)$  and

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \quad (12)$$

Then, if

$$\liminf_{r \rightarrow +\infty} \frac{\alpha(r)}{\sigma(r)} > 0, \quad (13)$$

the system (1) is ISS.

# Conclusions of Part I

- 1 The principle of adding exponential term in relaxed LKFs to make them strict, does not systematically work even for  $1D$  TDS.
- 2 Growth condition is proposed to conclude ISS with relaxed LKF.
- 3 The proposed condition turns out to extend the existing ones.
- 4 The conjecture is still an open question.

## PART II

# Rapid stabilization of general linear systems with F-equivalence

Joint work with [Amaury Hayat](#)

We consider the following linear PDE system

$$\partial_t u = \mathcal{A}u + Bw(t), \quad t \geq 0. \quad (14)$$

where  $\mathcal{A}$  and  $B$  are linear operator,  $u$  the state of the system and  $w$  the control variable.

## Objective

Provide conditions to stabilize the system (14), i.e

- ▶ Exhibit a feedback law  $K$  such that  $w = Ku(t, \cdot)$  and the closed-loop system

$$\partial_t u = \mathcal{A}u + BKu, \quad t \geq 0 \quad (15)$$

is well-posed;

- ▶ And show that any solution of (15) converges exponentially quick to 0, i.e

$$\|u(t, \cdot)\| \leq ke^{-\lambda t} \|u(0, \cdot)\|, \quad \forall t \geq 0. \quad (16)$$

# Description of the used method

Let  $\lambda > 0$ . Assume that  $\mathcal{A}$  generates a **dissipative  $C^0$  semi-group** and consider the following target system

$$\partial_t v = (\mathcal{A} - \lambda I)v, \quad t \geq 0. \quad (17)$$

There exists  $\omega \in \mathbb{R}$ ,  $k \geq 1$  such that the system (17) converges exponentially to 0, i.e

$$\|v(t, \cdot)\| \leq ke^{-(\lambda - \omega)t} \|v(0, \cdot)\|, \quad \forall t \geq 0. \quad (18)$$

Assume that there exists  $(T, K)$  such that  $v = Tu$  and (15) is well-defined.

## Remark 1

*If  $T$  is an isomorphism, we obtain*

$$\|u(t, \cdot)\| \leq Ke^{-(\lambda - \omega)t} \|u(0, \cdot)\|, \quad \forall t \geq 0, \quad (19)$$

*where  $K := k\|T^{-1}\|\|T\|$ .*

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## 1 Volterra transformations of the second kind

- [Balogh and Krstic, 2002, Bošković et al., 2003]
- [Krstic and Smyshlyaev, 2008]

## 2 Fredholm transformations

- Korteweg–de Vries equation [Coron and Lü, 2014]
- Kuramoto–Sivashinsky equation [Coron and Lü, 2015]
- Boundary-controlled systems  
[Coron et al., 2017, Deutscher and Gabriel, 2019, Redaud et al., 2022]
- The linear Schrodinger equation [Coron et al., 2018]
- Degenerate parabolic equations [Gagnon et al., 2021, Lissy and Moreno, 2023]
- The transport equation [Zhang, 2022]
- The linearized Saint-Venant system [Coron et al., 2022]
- The 1D heat equation [Gagnon et al., 2022b]
- The skew-adjoint system [Gagnon et al., 2022a]

# Question?

## Conjecture 2

*Assume that the pair  $(\mathcal{A}, B)$  is **exactly** controllable and admissible in a certain Hilbert space  $H$ . Then, for any  $\lambda > 0$  there exists a unique isomorphism-feedback pair  $(T, K)$  such that  $T$  transforms the system (15) into the target system (17). In particular any solution of the closed-loop system (15) converges exponentially to zero.*

# Answer in finite dimension

[Coron, 2015]

## Theorem 8 (F-equivalence in finite dimension)

Consider a linear finite dimension system defined as

$$\dot{y} = Ay + Bz, \quad (20)$$

where  $y \in \mathbb{R}^n$ ,  $z \in \mathbb{R}$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ . Assume that the pair  $(A, B)$  is controllable, i.e

$$\text{rank}\{B, A^2B, \dots, A^{n-1}B\} = n. \quad (21)$$

Then, for any  $\lambda > 0$ , there exists a unique isomorphism-feedback pair  $(T, K) \in GL(n, \mathbb{R}) \times \mathbb{R}^{1 \times n}$  such that  $T$  transforms the system

$$\dot{y} = Ay + BKy,$$

into the exponentially stable system

$$\dot{y} = Ay - \lambda y.$$

# Answer in infinite dimension?

- 1 Several results in the literature investigated the question
  - [Gagnon et al., 2022b] where  $\mathcal{A}$  is self-adjoint
  - [Gagnon et al., 2022a] when  $\mathcal{A}$  is a skew-adjoint
- 2 Can we extend these results to the framework of any "**spectral**" linear operator  $\mathcal{A}$ ?

Assume that

- 1  $\mathcal{A}$  generates a dissipative  $C^0$  semi-group on a Hilbert space  $X$ .
- 2 The eigenvectors  $\varphi_n$  of  $\mathcal{A}$  form a Riesz basis of  $X$ .
- 3 The eigenvalues  $\lambda_n$  have finite multiplicity and there exists  $\alpha > 1$  such that

$$n^\alpha \lesssim |\lambda_n| + 1 \lesssim n^\alpha, \quad \forall n \in \mathbb{N}^*, \quad (22)$$

$$|\lambda_n - \lambda_p| \gtrsim n^{\alpha-1} |n - p|, \quad \forall n, p \in \mathbb{N}^*. \quad (23)$$

## Theorem 9

Let  $\gamma \in [0, (\alpha - 1)/2)$ . If  $B$  satisfies

$$c_1 \leq |\langle B, \widetilde{\varphi}_n \rangle| \leq c_2 n^\gamma, \quad \forall n \in \mathbb{N}^*, \quad (24)$$

Then, there exist a feedback  $K$  and an isomorphism  $T$  such that  $T$  maps the system

$$\partial_t u = \mathcal{A}u + BKu \quad (25)$$

to the system

$$\partial_t v = \mathcal{A}v - \lambda v. \quad (26)$$

For any  $\mu > 0$ , (25) is exponentially stable with decay rate  $\mu > 0$ .

$(\widetilde{\varphi}_n)_n$  is the orthogonal basis of  $(\varphi_n)_n$  as  $\langle \varphi_n, \widetilde{\varphi}_m \rangle = \delta_{nm}$ .

## Theorem 10 (Rapid stabilization in $X$ )

Let  $\gamma \in [0, (\alpha - 1)/2)$ ,  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$ , and  $B \in \mathcal{H}^{-\alpha}$  such that

$$c_1 n^r \leq |\langle B, \tilde{\varphi}_n \rangle| \leq c_2 n^{r+\gamma}, \quad (27)$$

then for any  $\mu > 0$ , there exists a (constructive) bounded linear feedback  $K_\gamma$  such that the system (15) is exponentially stable in the study space  $X$  with decay rate  $\mu$ .

## Conclusions of Part II

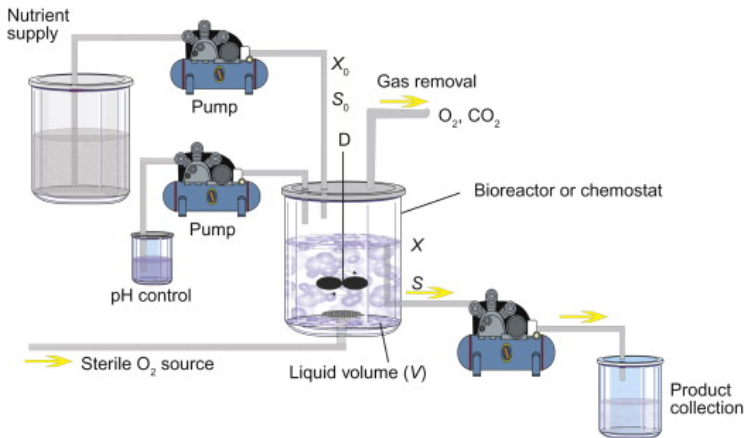
- 1 We provide more relaxed condition to stabilize infinite dimensional system which involves a general spectral operator
- 2 The obtained conditions are weak controllability requirement and making our result strong
- 3 More investigations are needed to handle the case of infinite multiplicity of eigenvalues.

## PART III

### The simplest chemostat model

Joint work with [Iasson Karafyllis](#), [Miroslav Krstic](#) and [Antoine Chaillet](#).

# The chemostat: A microbial bioreactor



The chemostats are used to

- 1 produce antibiotics and some proteins
- 2 treat water (degradation of pollutant)
- 3 produce biomass
- 4 ...

The simplest chemostat is modeled by:

$$\begin{cases} \dot{X} = (p_0\mu(S) - b - D)X \\ \dot{S} = D(S_{\text{in}} - S) - \mu(S)X. \end{cases} \quad (28)$$

Symbols	Amounts
$X$	Microbial concentration or biomass
$S$	Concentration of the substrate (nutrient)
$S_{\text{in}}$	Inlet concentration of the nutrient
$\mu(S)$	Specific growth rate
$D$	Dilution rate
$b$	Mortality rate
$p_0 = 1/k$	Yield factor

The parameters  $X, S, D, b, p_0$  are positive and  $\mu \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$  with  $\mu(0) = 0$ .

The goal in the chemostat is to maintain the system in a **non washout** steady state.

# Equilibrium point

An equilibrium point of the chemostat model (28) is  $(X^*, S^*)$  such that

$$p_0\mu(S^*) = b + D^*,$$
$$X^* = \frac{D^*(S_{\text{in}} - S^*)}{\mu(S^*)}.$$

Given  $D^* > 0$ , it may exist many equilibrium points

Given  $D^* > 0$  and an equilibrium  $(X^*, S^*)$ , the linearization of the system (28) is

$$\begin{pmatrix} \dot{X} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} 0 & p_0 \mu'(S^*) X^* \\ -(b + D^*)/p_0 & -D^* - \mu'(S^*) X^* \end{pmatrix} \begin{pmatrix} X \\ S \end{pmatrix}. \quad (29)$$

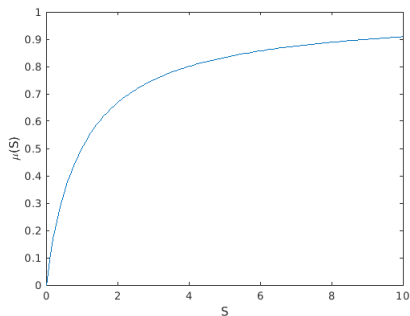
The characteristic polynomial is:

$$P(s) = s^2 + (D^* + \mu'(S^*) X^*) s + (b + D^*) \mu'(S^*) X^*. \quad (30)$$

The system is unstable if  $\mu'(S^*) < 0$ .

When considering the Monod kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S}, \quad (31)$$

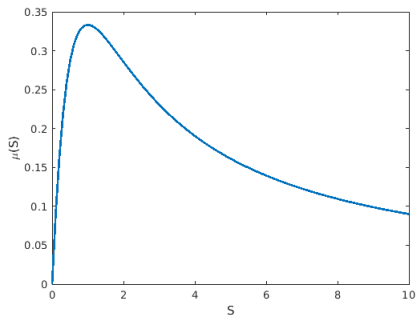


**Theorem 11** ([Dali-Youcef et al., 2022])

There exists a *unique positive equilibrium point* of (28) which is globally asymptotically stable and exponentially stable.

When, considering the Haldane kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S + aS^2}, \quad (32)$$



## Theorem 12

The system (28) admits two positive equilibrium points  $(X^*, S^*)$ ,  $(X^{**}, S^{**})$  such that

- ▶  $(X^*, S^*)$  is unstable.
- ▶  $(X^{**}, S^{**})$  is exponentially stable.

# Unstable open-loop chemostat model

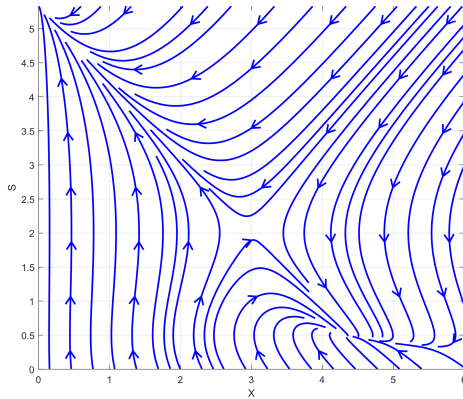


Figure: The phase diagram for the open-loop chemostat model (28) with Haldane equation

## Theorem 13

Assume that

$$p_0\mu(S) > b, \quad \forall S \in [S^*, S_{in}]. \quad (33)$$

Then for every  $\delta > 0$  and  $\alpha \in [0, 1)$  the feedback law defined as

$$D(X, S) = \frac{D^*\mu(S)X}{\mu(S^*)X^*} + \frac{\delta b}{(\mu(S^*))^{1+\alpha}} \begin{cases} |\mu(S) - \mu(S^*)|^{1+\alpha}, & \text{if } S \leq S^* \\ 0, & \text{if } S > S^* \end{cases} \quad (34)$$

achieves global asymptotic stabilization of  $(X^*, S^*)$ . Moreover, if  $\alpha > 0$  then the feedback law (34) also achieves local exponential stabilization of  $(X^*, S^*)$ .

# Stable closed-loop model

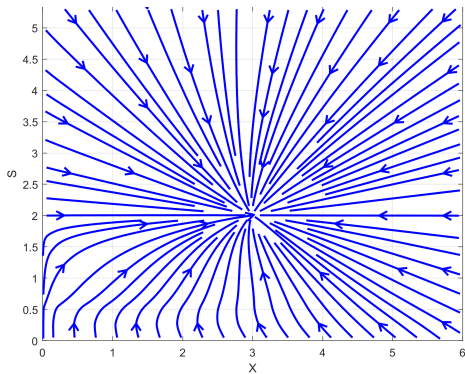


Figure: The phase diagram for the closed-loop chemostat model (28) with (34).

Is the sufficient condition (33) also necessary to ensure  
GAS of system (28)?

## Theorem 14

Assume that there exists a constant  $\bar{S} \in (S^*, S_{in})$  such that

$$p_0\mu(\bar{S}) < b \quad (35)$$

$$\mu'(S) \leq 0, \quad \forall S \in [\bar{S}, S_{in}]. \quad (36)$$

Then, there is no feedback law  $D(X, S) \geq 0$  that achieves global stabilization of the equilibrium  $(X^*, S^*)$  of (28).

# Perspectives

- 1 Allow the mortality  $b$  to be dependent of time  $t$
- 2 Consider  $S_{in}$  and/or the volume  $V$  to be dependent of  $t$
- 3 Consider another reproduction rate  $\mu$  than Monod and Haldane

Thank you for your attention



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