Building coercive Lyapunov-Krasovskii functional based on Razumikhin and Halanay approaches

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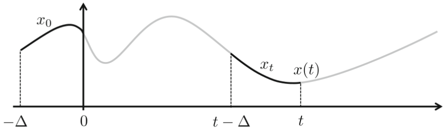
Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \tag{1}$$

$u(t) \in \mathbb{R}^m$ is the input
$x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the solution's history defined by $x_t(s) = x(t+s)$ for all $s \in [-\Delta, 0]$,



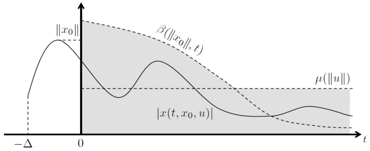
Δ is the maximum delay and we assume that f is Lipschitz on bounded set in what follows.

ISS and exp-ISS

Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$ and all $u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$



If $\beta(s,t) = kse^{-\lambda t}$, the TDS (1) is said to be **exponentially input-to-state stable (exp-ISS)**.

Lyapunov-Krasovskii-Functional (LKF)

Definition 3

A functional $V : C([-\Delta, 0], \mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|). \tag{3}$$

If V satisfies

$$\underline{\alpha}(\|x_t\|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|), \tag{4}$$

the LKF is said to be **coercive**.

$|x(t)|$ is a $\mathbb{R}^n$ norm which differs from $\|x_t\| := \sup_{\tau \in [-\Delta, 0]} |x_t(\tau)|$.

ISS from LKF

[Karafyllis et al., 2008]

Theorem 4

TDS (1) is ISS if and only if it admits an ISS LKF or coercive ISS LKF and there exist $\alpha, \gamma \in \mathcal{K}_\infty$ *such that*

$$D^+V \leq -\alpha(V(x_t)) + \gamma(|u(t)|). \tag{5}$$

[Teel, 1998]

Theorem 5

Let $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ *p. d and radially unbounded. If*

$$V_0(x(t)) \geq \max\left\{\rho\left(\max_{s\in[-\Delta,0]} V_0(x_t(s))\right), \gamma(|u(t)|)\right\} \Rightarrow \dot{V}_0(x(t)) \leq -\alpha(|x(t)|), \quad (6)$$

with $\rho(s) < s$ *for all* $s > 0$, *then the TDS* (1) *is ISS.*

Motivation!

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}. \quad (7)$$

Using standard manipulations, we were not able to provide an explicit ISS LKF to ensure ISS of TDS (7).

[Loko et al., 2024b](CDC)

Proposition 1

System (7) ***is ISS*** *using Razumikhin Theorem's.*

Our goal

Is there a bridge between these approaches?

How can we construct an explicit ISS LKF from Razumikhin or Halanay function?

About linear TDS: LMI's condition

$$\dot{x}(t) = Ax(t) + Bx(t-\Delta). \tag{8}$$

From [Gu et al., 2003], the TDS (8) is AS if there exist s.p.d matrix P and $c > 0$ as

$$\begin{pmatrix} A^\top P + PA + cP & PB \\ B^\top P & -cP \end{pmatrix} \prec 0. \tag{9}$$

[Loko et al., 2024a]

Proposition 2

Under condition (9)*, the functional*

$$V(x_t) := \max_{s\in[-\Delta,0]} e^{cs} x_t(s)^\top P x_t(s)$$

is a coercive LKF for TDS (8)*.*

About nonlinear TDS

Theorem 6

Let V_0 satisfies Razumikhin condition (6) *with*

$$\rho_0 := \sup_{s>0} \frac{\rho(s)}{s} < 1.$$

Then the functional

$$V(x_t) := \max_{s\in[-\Delta,0]} \rho_0^{-\tau/\Delta} V_0(x_t(s))$$

is a coercive ISS LKF of (1).

Application: Chemical reactor model

Consider the following chemical reactor model: $c,\xi,\mu > 0$

$$\partial_t w(t,z) + c\partial_z w(t,z) = -\xi w(t,z) + \xi x(t) \tag{10a}$$

$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu \int_0^1 w(t,z)dz + u(t). \tag{10b}$$

$$w(t,0) = 0, \quad \forall t \geq 0. \tag{10c}$$

[Loko et al., 2024a]

Proposition 3

Assume that

$$\sup_{x\neq 0} \frac{|g(x)|}{|x|} < 1 + \frac{\mu c}{\xi}\left(1 - e^{-\xi/c}\right). \tag{11}$$

Then the system (10) *is exp-ISS and there exist* $k,\lambda,\gamma_0 > 0$ *such that, its solution satisfies*

$$\|X(t)\| \leq k\|X(0)\|e^{-\lambda t} + \gamma_0\|u_{[0,t]}\|, \quad \forall t \geq 0.$$

where $X(t) = (w(t,\cdot),x(t))$.

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Comments

The present result extends the stability result in [Karafyllis and Krstic, 2021] in the following ways:

1. The restriction (11) is less demanding
2. We do not simply prove exponential stability but the much stronger exp-ISS property with respect to the input u.
3. An explicit ISS Lyapunov functional is constructed, so one can study the ISS property with respect to boundary disturbances or disturbances acting in the PDE.

ISS from Halanay

[Pepe, 2021]

Theorem 7

Let $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ p. d and radially unbounded. If

$$\dot{V}_0(x(t)) \leq -\alpha(|x(t)|) + \rho\left(\max_{s\in[-\Delta,0]} V_0(x_t(s))\right) + \gamma(|u(t)|), \tag{12}$$

with $\alpha - \rho \in \mathcal{K}_\infty$, then the TDS (1) is ISS.

Conclusion

1. Coercive ISS LKF can be constructed from Razumikhin and Halanay aproaches.
2. Stability condition for chemical reactor model is improved and its explicit solution estimate is provided.
3. What about the vector version of Halanay?

Discussion in the break!

For those who are intersted, I also get some other works:

1. E. Loko, A. Chaillet, Y. Wang, I. Karafyllis, P. Pepe (2024).
   Growth conditions to ensure input-to-state stability under point-wise dissipation.
   Control on Decision Conference: (under review)
2. A. Hayat and E. Loko (2024).
   Fredholm backstepping and rapid stabilization of general linear systems.
   In process

Thank you for your attention!

Gu, K., Chen, J., and Kharitonov, V. L. (2003).
Stability of time-delay systems.
Springer Science & Business Media.

Karafyllis, I. and Krstic, M. (2021).
Input-to-state stability for PDEs.
Encyclopedia of Systems and Control, pages 1030–1033.

Karafyllis, I., Pepe, P., and Jiang, Z.-P. (2008).
Input-to-output stability for systems described by retarded functional differential equations.
European Journal of Control, 14(6):539–555.

Loko, E., Chaillet, A., and Karafyllis, I. (2024a).
Building coercive Lyapunov–Krasovskii functionals based on Razumikhin and Halanay approaches.
International Journal of Robust and Nonlinear Control.

Loko, E., Chaillet, A., Wang, Y., Karafyllis, I., and Pepe, P. (2024b).
Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation. In process.

Pepe, P. (2021).

A nonlinear version of Halanay's inequality for the uniform convergence to the origin.
Mathematical Control and Related Fields.

Teel, A. R. (1998).
Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem.
IEEE Transactions on Automatic Control, 43(7):960–964.