

# GROWTH CONDITIONS TO ENSURE INPUT-TO-STATE STABILITY OF TIME-DELAY SYSTEMS WITH POINT-WISE DISSIPATION

Epiphane Loko

Univ Paris Saclay, CNRS, CentraleSupélec, L2S- ENPC, CERMICS

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Joint work with Antoine Chaillet, Yuan Wang, Iason Karafyllis, and Pierdomenico Pepe

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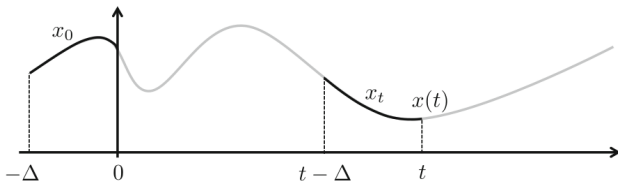
# Time-delay system

## Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \quad (1)$$

$u \in L_{loc}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ ,  $\Delta > 0$ ,  $x_t : [-\Delta, 0] \rightarrow \mathbb{R}^n$ ,  $x_t(s) = x(t+s)$ ,  $\forall s \in [-\Delta, 0]$ . Assume  $f$ , Lipschitz on bounded sets and  $f(0, 0) = 0$ , in what follows.



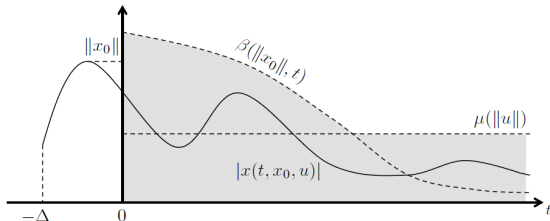
# Input-to-state stability (ISS)

## Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist  $\beta \in \mathcal{KL}$  and  $\mu \in \mathcal{K}_\infty$  such that, for all  $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$  and all  $u \in L_{loc}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ ,

$$|x(t, x_0, u)| \leq \beta(\|x_0\|, t) + \mu(\|u_{[0,t]}\|), \quad \forall t \geq 0. \quad (2)$$

$$|x(t, x_0, u)| \leq \max\{\beta(\|x_0\|, t), \mu(\|u_{[0,t]}\|)\}, \quad \forall t \geq 0.$$



## Definition 3

A functional  $V : C([-Δ, 0], \mathbb{R}^n) \rightarrow \mathbb{R}_{\geq 0}$  is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_{\infty}$  such that, for all  $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \bar{\alpha}(\|x_t\|). \quad (3)$$

$$|x(t)| := \left( \sum_{i=1}^n |x_i(t)|^2 \right)^{1/2} \quad \text{and} \quad \|x_t\| := \sup_{\tau \in [-\Delta, 0]} |x_t(\tau)|.$$

## Definition 4

For TDS (1), an LKF  $V$  is said to be

- 1 an ISS LKF with **LKF-wise** dissipation if there exist  $\alpha, \gamma \in \mathcal{K}_\infty$  such that

$$D^+V \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad (4)$$

- 2 an ISS LKF with **point-wise** dissipation if there exist  $\alpha, \gamma \in \mathcal{K}_\infty$  such that

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \quad (5)$$

$D^+V$  is the Driver's derivative of the functional  $V$  along the solution of (1).

Iason Karafyllis, Pierdomenico Pepe, and Zhong-Ping Jiang (2008). “Input-to-output stability for systems described by retarded functional differential equations”. In: *European Journal of Control* 14.6, pp. 539–555

## Theorem 5

*TDS (1) is ISS if and only if it admits an ISS LKF with **LKF-wise dissipation**.*

Is the point-wise dissipation enough to ensure ISS of TDS?

Chaillet, Pepe, Mason, and Chitour 2017

## Conjecture 1

*Assume that the system (1) admits a LKF with a point-wise dissipation. Then it is ISS.*

In response,

- 1 Antoine Chaillet, Pierdomenico Pepe, Paolo Mason, and Yacine Chitour (2017). “Is a point-wise dissipation rate enough to show ISS for time-delay systems?” In: *IFAC-PapersOnLine* 50.1, pp. 14356–14361
- 2 Antoine Chaillet, Iasson Karafyllis, Pierdomenico Pepe, and Yuan Wang (2023). “Growth conditions for global exponential stability and exp-ISS of time-delay systems under point-wise dissipation”. In: *Systems & Control Letters* 178, p. 105570

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## A way to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^0 V_2(x_t(s)) ds, \quad (6)$$

which dissipates point-wisely as follow

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding  $ke^{cs}$  in the kernel of integral part of (6), we get the LKF

$$W(x_t) = V_1(x(t)) + \int_{-\Delta}^0 ke^{cs} V_2(x_t(s)) ds.$$

### Proposition 1

If  $\alpha(|x(t)|) \geq pV_2(x(t))$ , ( $p > 0$ ) then  $W$  is a LKF with LKF-wise dissipation and the TDS (1) is ISS.

Does this trick work systematically?

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \quad (7)$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^0 x_t(s)^4 ds,$$

$$W(x_t) := \frac{1}{4}x(t)^4 + \int_{-\Delta}^0 ke^{cs} x_t(s)^4 ds.$$

## Proposition 2

- 1 System (7) **is ISS**.
- 2  $V$  is an ISS LKF with **point-wise dissipation** for (7)
- 3 Given any  $k, c > 0$ ,  $W$  **is not a LKF with LKF-wise dissipation**.

## Sketch of the proof

Here  $\Delta = 1$ , we set  $\mathcal{X} := C^0([-1, 0], \mathbb{R})$  and we denote by  $f$  the dynamic of (7).

$$W \text{ dissipates LKF-wise} \Leftrightarrow D^+ W(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|), \quad \forall \phi \in \mathcal{X}, v \in \mathbb{R}.$$

In particular for  $v = 0$ , this implies

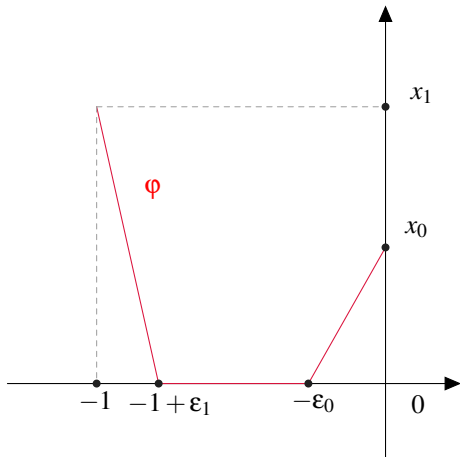
$$W \text{ dissipates LKF-wise} \Rightarrow D_0^+ W := D^+ W(\phi, f(\phi, 0)) \leq -\alpha(V(\phi)) \leq 0, \quad \forall \phi \in \mathcal{X}.$$

# Sketch of the proof

Set  $x_0 := \phi(0)$ ,  $x_1 := \phi(-1)$ .

$$D_0^+ W = x_0^4 \left( k - 1 - \frac{1}{1+x_0^2} \right) + x_1^4 \left( \frac{x_0^3}{1+|x_0|^3} - ke^{-c} \right) - kc \int_{-1}^0 e^{cs} \phi(s)^4 ds.$$

$$D^+ W(\varphi, f(\varphi, 0)) > 0.$$



## Theorem 6

Assume that there exists a LKF  $V$  which dissipates *point-wisely* as follow,

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \quad (8)$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \quad (9)$$

Then, if

$$\liminf_{r \rightarrow +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \quad (10)$$

the system (1) is ISS.

## Corollary 7

Assume that there exists a LKF  $V$  which dissipates *point-wisely* as follow,

$$D^+V \leq -Q(x(t)) + \gamma(|u(t)|). \quad (11)$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \quad (12)$$

Then, if

$$\liminf_{r \rightarrow +\infty} \frac{r}{\sigma(r)} > 0, \quad (13)$$

the system (1) is ISS.

# Conclusion

- 1 The principle of adding exponential term in relaxed LKFs to make them strict, does not systematically work even for  $1D$  TDS.
- 2 Growth condition is proposed to conclude ISS with relaxed LKF.
- 3 The proposed condition turns out to extend the existing ones.
- 4 The conjecture is still an open question.

Thank you for your attention!

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