Growth conditions to ensure input-to-state stability of time-delay systems with point-wise dissipation

Epiphane LOKO

dagbegnon.loko@enpc.fr/ dagbegnon-epiphane.loko@centralesupelec.fr

Université Paris Saclay [L2S] & Ecole Nationale des Ponts et chaussées [CERMICS]

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Supervisors: **Antoine CHAILLET and Amaury HAYAT**

Joint work with A. Chaillet, Y. Wang, I. Karafyllis, and P. Pepe

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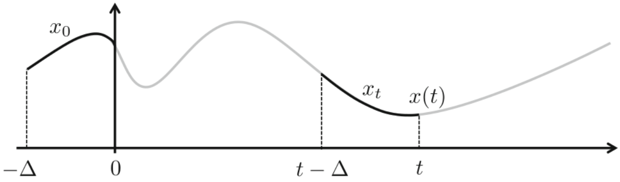
Time-delay system

Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \quad (1)$$

$u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$, $\Delta > 0$, $x_t : [-\Delta, 0] \to \mathbb{R}^n$, $x_t(s) = x(t+s), \forall s \in [-\Delta, 0]$. Assume f, Lipschitz on bounded sets and $f(0,0) = 0$, in what follows.



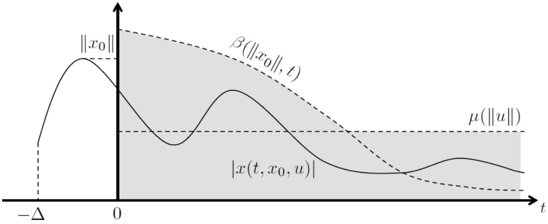
Input-to-state stability (ISS)

Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$ and all $u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$

$$|x(t, x_0, u)| \leq \max\{\beta(||x_0||, t), \mu(||u_{[0,t]}||)\}, \quad \forall t \geq 0.$$



Lyapunov-Krasovskii-Functional (LKF)

Definition 3

A functional $V : C([-\Delta,0],\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|). \tag{3}$$

$$|x(t)| := \left(\sum_{i=1}^{n} |x_i(t)|^2 \right)^{1/2}$$ and $$\|x_t\| := \sup_{\tau \in [-\Delta,0]} |x_t(\tau)|.$$

Dissipations of LKF

Definition 4

For TDS (1), an LKF V is said to be

1. an ISS LKF with LKF-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \tag{4}$$

2. an ISS LKF with point-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{5}$$

D^+V is the Driver's derivative of the functional V along the solution of (1).

General ISS characterization

[Karafyllis et al., 2008]

Theorem 5

TDS (1) *is ISS if and only if it admits an ISS LKF with LKF-wise dissipation.*

Is a point-wise dissipation enough to ensure ISS of TDS?

[Chaillet et al., 2017]

Conjecture 1

Assume that the system (1) *admits a LKF with a point-wise dissipation. Then it is ISS.*

In response,

1. [Chaillet et al., 2017] provides a sufficient condition on the growth of the vector field of TDS.
2. [Chaillet et al., 2023] provides some growth conditions on the LKF to ensure exp-ISS.

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A way to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} V_2(x_t(s))ds, \tag{6}$$

which dissipates point-wisely as follow

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding ke^{cs} in the kernel of integral part of (6), we get the LKF

$$W(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

Proposition 1

If $\alpha(|x(t)|) \geq pV_2(x(t))$, $(p > 0)$ then W is a LKF with LKF-wise dissipation and the TDS (1) *is ISS.*

Corollary: $(V_1, V_2, \alpha$ quadractics)=[Orłowski et al., 2022, Lemma 1].

Does this trick work systematically?

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{7}$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^{0} x_t(s)^4 ds,$$

$$W(x_t) := \frac{1}{4}x(t)^4 + \int_{-\Delta}^{0} ke^{cs} x_t(s)^4 ds.$$

Proposition 2

1. *System* (7) ***is ISS**.*
2. *V is an ISS LKF with **point-wise dissipation** for* (7)
3. *Given any $k, c > 0$, W is not a LKF with LKF-wise dissipation.*

The ISS of (7) is shown using Razumikhin method ([Teel, 1998]).

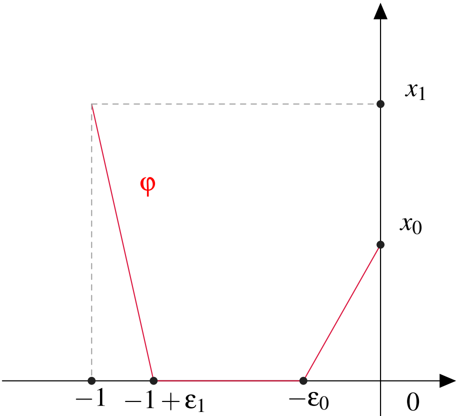
Sketch of the proof

Here $\Delta = 1$, we set $\mathcal{X} := C^0([-1,0],\mathbb{R})$ and we denote by f the dynamic of (7).

W dissipates LKF-wise $\Rightarrow$ $D_0^+ W := D^+ W(\phi, f(\phi,0)) \leq -\alpha(V(\phi)) \leq 0, \quad \forall \phi \in \mathcal{X}.$

$$D_0^+ W = x_0^4 \left(k - 1 - \frac{1}{1+x_0^2} \right) + x_1^4 \left(\frac{x_0^3}{1+|x_0|^3} - k e^{-c} \right) - kc \int_{-1}^{0} e^{cs} \phi(s)^4 ds.$$

$D^+ W(\varphi, f(\varphi, 0)) > 0.$



ISS under point-wise dissipation

Theorem 6

Assume that there exists a LKF V which dissipates point-wisely as follow,

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \tag{8}$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{9}$$

Then, if

$$\liminf_{r\to+\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{10}$$

the system (1) *is ISS.*

Corollary 7

Assume that there exists a LKF V which dissipates point-wisely as follow,

$$D^+V \leq -Q(x(t)) + \gamma(|u(t)|). \tag{11}$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{12}$$

Then, if

$$\liminf_{r\to+\infty} \frac{r}{\sigma(r)} > 0, \tag{13}$$

the system (1) *is ISS.*

Conclusion

1. The principle of adding exponential term in relaxed LKFs to make them strict, does not systematically work even for $1D$ TDS.
2. Growth condition is proposed to conclude ISS with relaxed LKF.
3. The proposed condition turns out to extend the existing ones.
4. The conjecture is still an open question.

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