Novel point-wise dissipation conditions for input-to-state stability of time-delay systems

Epiphane LOKO

dagbegnon.loko@enpc.fr/ dagbegnon-epiphane.loko@centralesupelec.fr

Université Paris Saclay & Ecole des Ponts ParisTech

L2S & CERMICS

Supervisors:

Antoine CHAILLET
Amaury HAYAT

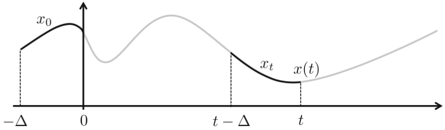
May 21, 2024

Definition 1

A **nonlinear time-delay system** is a system modeled by functional differential equation of the type:

$$\dot{x}(t) = f(x_t, u(t)), \quad (1)$$

where $u(t) \in \mathbb{R}^m$ is the input, and $x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the solution's history defined by $x_t(s) = x(t+s)$ for all $s \in [-\Delta, 0]$, where $\Delta \geq 0$ denotes the maximum time delay involved.



Assumption: the vector field f is **Lipschitz on bounded sets** and satisfies

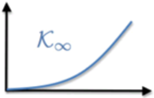
$$f(0,0) = 0$$

we denote by $\mathcal{X}^n$ the set of continuous functions maping $[-\Delta, 0]$ into $\mathbb{R}^n$ i.e

$$\mathcal{X}^n = \mathcal{C}([-\Delta, 0], \mathbb{R}^n)$$

Comparision functions

1. Class $\mathcal{K}_\infty$ function: zero at zero, continuous, increasing and unbounded.
2. Class $\mathcal{KL}$ function : $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
    - $\beta(\cdot, t)$ is zero at zero, continuous and increasing
    - $\beta(s, \cdot)$ is continuous, decreasing and vanishing at infinity.



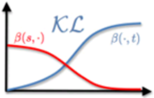


Figure: Class $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

ISS and exp-ISS

Definition 2

System (1) is said to be **input-to-state stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$,

$$|x(t,x_0,u)| \leq \beta(||x_0||,t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$

In particular, when there exist $k, \lambda \geq 0$, such that β is defined as:

$$\beta(s,t) = kse^{-\lambda t}, \ \forall s,t \geq 0 \tag{3}$$

then (1) is said to be **exponentially input-to-state stable (exp-ISS)**.

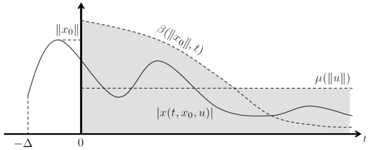


Figure: Schematic representation of the time evolution of an ISS system's solution.

ISS properties

ISS automatically guarantees the following properties:

1. **0-GAS**: the input-free system $\dot{x}(t) = f(x_t, 0)$ is Globally Asymptotically Stable (GAS)
2. **Bounded Input-Bounded State (BIBS)** property: given any $x_0 \in \mathcal{X}^n$ and any $u \in \mathcal{U}^m$,
$$||u|| < +\infty \quad \Rightarrow \quad \sup_{t \geq 0} |x(t, x_0, u)| < +\infty \tag{4}$$
3. **Asymptotic Gain (AG)** property: there exist $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$:
$$\limsup_{t \to +\infty} |x(t, x_0, u)| \leq \mu(||u||) \tag{5}$$
4. **Converging Input-Converging State (CICS)** property: given any $x_0 \in \mathcal{X}^n$ and any $u \in \mathcal{U}^m$,
$$\lim_{t \to +\infty} |u(t)| = 0 \quad \Rightarrow \quad \lim_{t \to +\infty} |x(t, x_0, u)| = 0 \tag{6}$$

General way to characterize ISS of time-delay systems: Lyapunov-Krasovskii functional

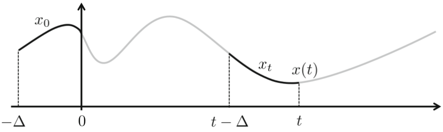
Definition 3

A functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{7}$$

$$\|\phi\| := \sup_{\tau \in [-\Delta, 0]} |\phi(\tau)|.$$

Actually, the function ϕ represents x_t.



Dissipations of LKF

Definition 4

For system (1), an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is said to be

1. an ISS LKF with **LKF-wise** dissipation if there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|) \tag{8}$$

2. an ISS LKF with **point-wise** dissipation or **relaxed ISS LKF** if there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|) \tag{9}$$

where (8) and (9) are meant to hold for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$.

- From (7), the following implication holds:

$$(8) \Rightarrow (9)$$

- The point-wise dissipation condition (9) turns out to be easier to obtain as it involves the current norm $|\phi(0)|$ of solution and not the functional V itself.
- ISS of time-delay systems can be completely estabilished by the LKF V with LKF-wise dissipation (8):

[Karafyllis et al., 2008]

Theorem 5

System (1) *is ISS if and only if it admits an ISS LKF with LKF-wise dissipation.*

What happens from LKF with point-wise dissipation?

[Chaillet et al., 2023b, Conjecture 4]

Conjecture 1

Assume that there exist an LKF V with point-wise dissipation meaning (9) *holds. Then the system* (1) *is ISS.*

Conjecture 1 is still an open question. In response,

1. additional assumptions on the growth of the LKF or the vector field itself are stated in [Chaillet et al., 2017].
2. growth conditions are also provided in [Chaillet et al., 2023a, Theorems 3 $\&$ 4] to ensure exponential ISS with point-wise dissipation on LKF.

Theorem 6

Assume that there exist a functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *which is Lipschitz on bounded sets,* $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ *and* $Q : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ *continuously differentiable, positive definite and radially unbounded such that,*

$$\begin{aligned} &\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|) \\ &D^+V(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|) \end{aligned} \tag{10}$$

Assume further that there exists a function $\sigma \in \mathcal{K}_\infty$ *such that*

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma\left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))\right) + \gamma(|v|). \tag{11}$$

Then, under the condition that

$$\liminf_{r \to +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{12}$$

the system (1) *is ISS.*

"Strictification" of LKF

1. Let consider $V_1, V_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ positive definite and radially unbounded. Then

$$V(\phi) = V_1(\phi(0)) + \int_{-\Delta}^{0} V_2(\phi(s))ds, \quad \forall \phi \in \mathcal{X}^n \tag{13}$$

   is an LKF candidate.
2. A wide class of Lyapunov-Krasovskii functionals used in practice, is of the form (13).
3. It is usually more simple to obtain the point-wise dissipation with this LKF.
4. There exists a trick to "**strictify**" this LKF meaning we can build from it an LKF with LKF-wise dissipation.
5.

$$W(\phi) = V_1(\phi(0)) + \int_{-\Delta}^{0} ke^{cs} V_2(\phi(s))ds, \quad \forall \phi \in \mathcal{X}^n. \tag{14}$$

"Strictification" of LKF

Theorem 7

Let V *is an LKF of the form* (13), *i.e*

$$V(\phi) = V_1(\phi(0)) + \int_{-\Delta}^{0} V_2(\phi(s))ds, \quad \forall \phi \in \mathcal{X}^n$$

which dissipates point-wisely as in (10) *meaning*

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|).$$

If there exist $\varepsilon > 0$ *such that for all* $\phi \in \mathcal{X}^n$:

$$\alpha(Q(\phi(0))) \geq \varepsilon V_2(\phi(0)), \tag{15}$$

then there exist $c, k > 0$, *such that the functional* W *defined by*

$$W(\phi) = V_1(\phi(0)) + \int_{-\Delta}^{0} ke^{cs} V_2(\phi(s))ds, \quad \forall \phi \in \mathcal{X}^n. \tag{16}$$

is a strict ISS LKF for system (1).

Consider the following **one dimensional system**

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{17}$$

and the Lyapunov Krasovskii functionals (LKF) $V,\ W$ respectively defined as:

$$V(\phi) := \frac{1}{4}\phi(0)^4 + \int_{-1}^{0} \phi(s)^4 ds, \quad \forall\, \phi \in \mathcal{X}, \tag{18}$$

$$W(\phi) := \frac{1}{4}\phi(0)^4 + k\int_{-1}^{0} e^{cs}\phi(s)^4 ds, \quad \forall\, \phi \in \mathcal{X}, \tag{19}$$

where $c, k > 0$.
The LKF W is obtained by adding an exponential term with rate k in the kernel of integral term of V like in (16).

The following result shows that the LKF W will not be an strict LKF for system (17) whatever positive constants c and k are chosen.

Proposition 1

1. *System* (17) ***is ISS***.
2. *The functional V is an ISS LKF with* ***point-wise dissipation*** *for system* (17)
3. *For any positive constants c and k, the functional W* ***is not*** *LKF with LKF-wise dissipation.*

Razumikhin and Halanay approaches

1. There also exist less general ways to ensure ISS of (1) which are Razhumikhin and Halanay approaches.
2. These approaches are based on the Lyapunov functions which involve the current solution of the system.
3. Recently, we provide results to construct **coercive Lyapunov-Krasovskii functional** based on these approaches.

Conclusion

1. Two main reasons motivate the study of weather or not the relaxed ISS LKF allows to guarentee ISS for time-delay systems:
   - It is usually easier to estabilish point-wise dissipation without adding unnaturel exponential terms.
   - To be coherent to the original LKF theorem for GAS of time-delay systems without inputs.
2. Unfortunately, despite much investigations are done, the conjecture is still an open question.
3. The principle of adding exponential term in certain relaxed LKFs to make them strict, does not systematically work even for one dimensional systems.
4. A main future line of research would be to prove formally the conjecture

Thank you for your attention!

Chaillet, A., Karafyllis, I., Pepe, P., and Wang, Y. (2023a).
Growth conditions for global exponential stability and exp-iss of time-delay systems under point-wise dissipation.
Systems & Control Letters, 178:105570.

Chaillet, A., Karafyllis, I., Pepe, P., and Wang, Y. (2023b).
The iss framework for time-delay systems: a survey.
Mathematics of Control, Signals, and Systems, pages 1–70.

Chaillet, A., Pepe, P., Mason, P., and Chitour, Y. (2017).
Is a point-wise dissipation rate enough to show iss for time-delay systems?
IFAC-PapersOnLine, 50(1):14356–14361.
20th IFAC World Congress.

Karafyllis, I., Pepe, P., and Jiang, Z. (2008).
Input-to-output stability for systems described by retarded functional differential equations.
European Journal of Control, 14(6):539–555.