

# STABILIZATION OF CHEMOSTATS WITH MORTALITY AND SUBSTRATE DYNAMICS

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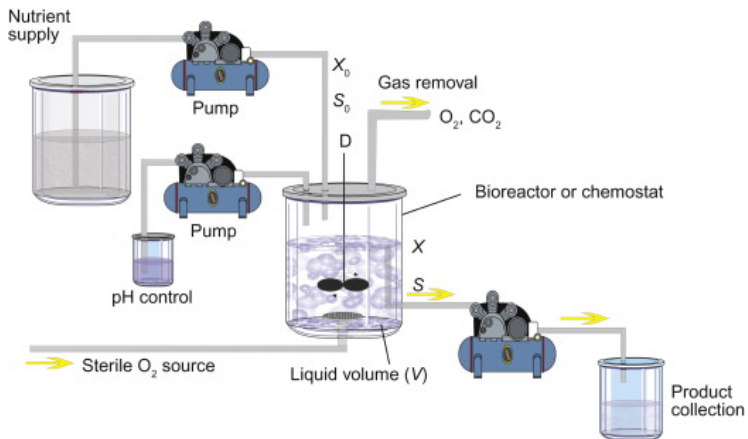
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## PART I

### The simplest chemostat model

# The chemostat: A microbial bioreactor



The chemostats are used to

- 1 produce antibiotics and some proteins
- 2 treat water (degradation of pollutant)
- 3 produce biomass
- 4 ...

The simplest chemostat is modeled by:

$$\begin{cases} \dot{X} = (p_0\mu(S) - b - D)X \\ \dot{S} = D(S_{\text{in}} - S) - \mu(S)X. \end{cases} \quad (1)$$

Symbols	Amounts
$X$	Microbial concentration or biomass
$S$	Concentration of the substrate (nutrient)
$S_{\text{in}}$	Inlet concentration of the nutrient
$\mu(S)$	Specific growth rate
$D$	Dilution rate
$b$	Mortality rate
$p_0 = 1/k$	Yield factor

The parameters  $X, S, D, b, p_0$  are positive and  $\mu \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$  with  $\mu(0) = 0$ .

The goal in the chemostat is to maintain the system in a **non washout** steady state.

# Equilibrium point

An equilibrium point of the chemostat model (1) is  $(X^*, S^*)$  such that

$$p_0\mu(S^*) = b + D^*,$$
$$X^* = \frac{D^*(S_{\text{in}} - S^*)}{\mu(S^*)}.$$

Given  $D^* > 0$ , it may exist many equilibrium points

Given  $D^* > 0$  and an equilibrium  $(X^*, S^*)$ , the linearization of the system (1) is

$$\begin{pmatrix} \dot{X} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} 0 & p_0 \mu'(S^*) X^* \\ -(b + D^*)/p_0 & -D^* - \mu'(S^*) X^* \end{pmatrix} \begin{pmatrix} X \\ S \end{pmatrix}. \quad (2)$$

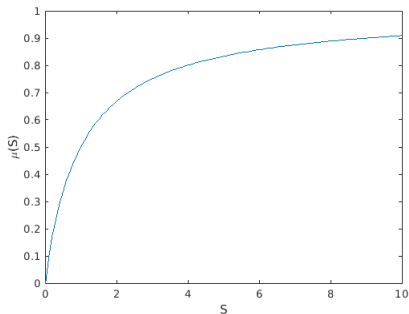
The characteristic polynomial is:

$$P(s) = s^2 + (D^* + \mu'(S^*) X^*) s + (b + D^*) \mu'(S^*) X^*. \quad (3)$$

The system is unstable if  $\mu'(S^*) < 0$ .

When considering the Monod kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S}, \quad (4)$$

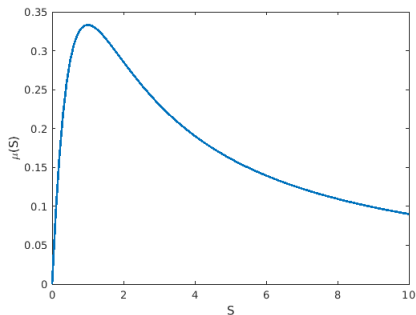


**Theorem 1 (Dali-Youcef, Rapaport, and Sari 2022)**

*There exists a **unique positive equilibrium point** of (1) which is globally asymptotically stable and exponentially stable.*

When, considering the Haldane kinetics

$$\mu(S) := \frac{\mu_{\max} S}{K + S + aS^2}, \quad (5)$$



## Theorem 2

The system (1) admits two positive equilibrium points  $(X^*, S^*)$ ,  $(X^{**}, S^{**})$  such that

- ▶  $(X^*, S^*)$  is unstable.
- ▶  $(X^{**}, S^{**})$  is exponentially stable.

# Unstable open-loop chemostat model

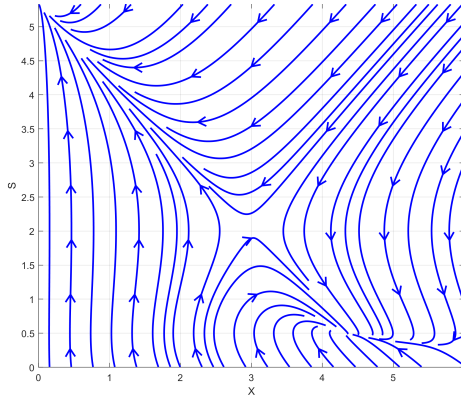


Figure: The phase diagram for the open-loop chemostat model (1) with Haldane equation

## Theorem 3

Assume that

$$p_0\mu(S) > b, \quad \forall S \in [S^*, S_{in}]. \quad (6)$$

Then for every  $\delta > 0$  and  $\alpha \in [0, 1)$  the feedback law defined as

$$D(X, S) = \frac{D^*\mu(S)X}{\mu(S^*)X^*} + \frac{\delta b}{(\mu(S^*))^{1+\alpha}} \begin{cases} |\mu(S) - \mu(S^*)|^{1+\alpha}, & \text{if } S \leq S^* \\ 0, & \text{if } S > S^* \end{cases} \quad (7)$$

achieves global asymptotic stabilization of  $(X^*, S^*)$ . Moreover, if  $\alpha > 0$  then the feedback law (7) also achieves local exponential stabilization of  $(X^*, S^*)$ .

# Stable closed-loop model

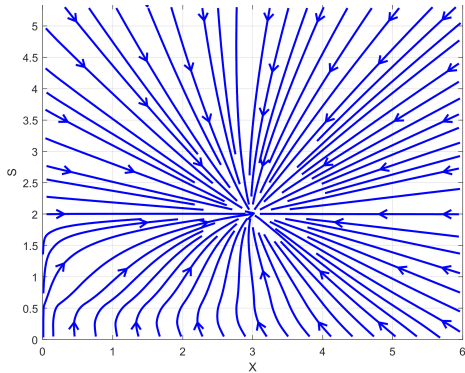


Figure: The phase diagram for the closed-loop chemostat model (1) with (7).

Is the sufficient condition (6) also necessary to ensure GAS of system (1)?

## Theorem 4

Assume that there exists a constant  $\bar{S} \in (S^*, S_{in})$  such that

$$p_0\mu(\bar{S}) < b \quad (8)$$

$$\mu'(S) \leq 0, \quad \forall S \in [\bar{S}, S_{in}]. \quad (9)$$

Then, there is no feedback law  $D(X, S) \geq 0$  that achieves global stabilization of the equilibrium  $(X^*, S^*)$  of (1).

## PART II

An age structured chemostat model with substrate dynamics

Consider the following age structured chemostat model with substrate dynamics

$$\frac{\partial f}{\partial t}(t, a) + \frac{\partial f}{\partial a}(t, a) = -(\beta(a) + D(t))f(t, a), \quad (10)$$

$$f(t, 0) = \mu(S(t)) \int_0^{+\infty} k(a)f(t, a)da, \quad (11)$$

$$\dot{S}(t) = D(t)(S_{\text{in}} - S(t)) - \mu(S(t)) \int_0^{+\infty} q(a)f(t, a)da, \quad (12)$$

where  $f(t, a) > 0$  is the distribution function of the microbial population in the chemostat at time  $t \geq 0$  and age  $a \geq 0$ .

We derived the following ODE chemostat model

$$\dot{S} = D(S_{\text{in}} - S) - \mu(S)X \quad (13a)$$

$$\dot{X} = q_0\mu(S)Y - (b + D)X \quad (13b)$$

$$\dot{Y} = p_0\mu(S)Y + \gamma X - (b + D)Y, \quad (13c)$$

where

$$X(t) = \int_0^{+\infty} q(a)f(t, a)da, \quad Y(t) = \int_0^{+\infty} k(a)f(t, a)da. \quad (14)$$

# Unstable open-loop equilibrium

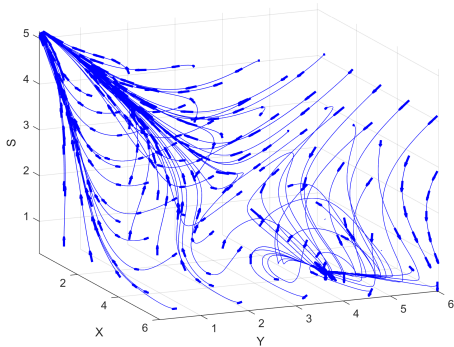


Figure: The phase diagram for the chemostat model (13) with Haldane kinetics.

**(A)** There exists a constant  $\varphi > 1$  such that for  $S \in [S^*, S_{in}]$  :

$$\left(b - \frac{\lambda(b + D^*)}{2}\right) \left(\frac{\mu(S^*)}{\mu(S)} - 1\right) + (1 + \lambda) \frac{\lambda^2 \varphi (b + D^*)^2}{4D^*} \left(\frac{\mu(S^*)}{\mu(S)} - 1\right)^2 < D^* \left(1 - \frac{1}{4(1 + \lambda)\varphi}\right)$$

## Theorem 5

*Under assumption (A) the feedback law*

$$D(X, S) = \frac{D^* \mu(S) X}{\mu(S^*) X^*} + \frac{\delta}{(\mu(S^*))^{1+\alpha}} \begin{cases} |\mu(S) - \mu(S^*)|^{1+\alpha}, & \text{if } S \leq S^* \\ 0, & \text{if } S > S^* \end{cases} \quad (15)$$

*achieves global asymptotic stabilization of the equilibrium. Moreover, if  $\alpha > 0$ , then the feedback law (15) achieves in addition local exponential stabilization of the equilibrium.*

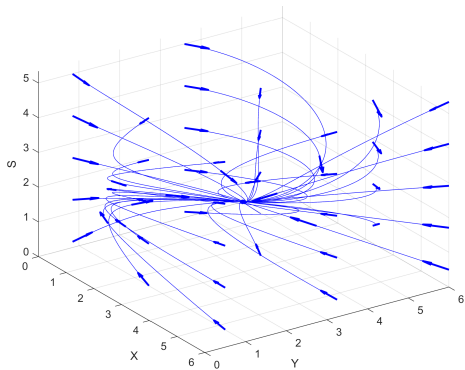


Figure: The phase diagram of the closed-loop system (13) with (15)

- 1 Consider  $S_{in}$  and/or the volume  $V$  to be depending on  $t$
- 2 Consider another reproduction rate  $\mu$  than Monod and Haldane
- 3 Investigate the stabilization of the PDE-ODE model (10)-(11)-(12)

Thank you for your attention