Rapid stabilization of general linear PDE and ISS of TDS under point-wise dissipation

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PART I

Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation

Joint work with
Antoine Chaillet, Yuan Wang, Iasson Karafyllis, and Pierdomenico Pepe

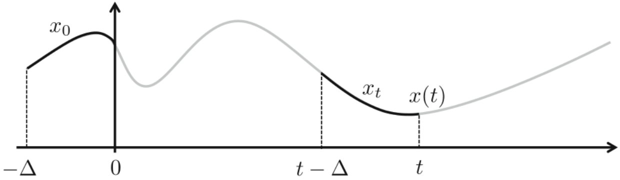
Time-delay system

Definition 1

A **time-delay system** (TDS) is a system modeled by:

$$\dot{x}(t) = f(x_t, u(t)). \tag{1}$$

$u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$, $\Delta > 0$, $\quad x_t : [-\Delta, 0] \to \mathbb{R}^n$, $\quad x_t(s) = x(t+s), \forall s \in [-\Delta, 0]$. Assume f, Lipschitz on bounded sets and $f(0,0) = 0$, in what follows.



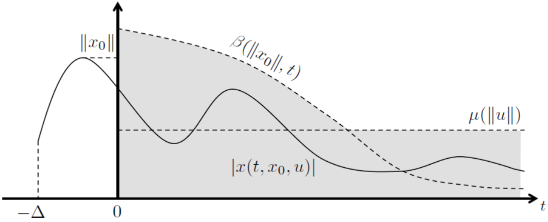
Input-to-state stability (ISS)

Definition 2

TDS (1) is said to be **input-to-state stable (ISS)** if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in C([-\Delta, 0], \mathbb{R}^n)$ and all $u \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^m)$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2}$$

$$|x(t, x_0, u)| \leq \max\{\beta(||x_0||, t), \mu(||u_{[0,t]}||)\}, \quad \forall t \geq 0.$$



Lyapunov-Krasovskii-Functional (LKF)

Definition 3

A functional $V : C([-\Delta,0],\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ is said to be a **Lyapunov-Krasovskii functional candidate (LKF)** if it is **Lipschitz on bounded sets** and there exist $\underline{\alpha},\overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $t \geq 0$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|). \tag{3}$$

$$|x(t)| := \left(\sum_{i=1}^{n} |x_i(t)|^2\right)^{1/2} \text{ and } \|x_t\| := \sup_{\tau\in[-\Delta,0]} |x_t(\tau)|.$$

Dissipations of LKF

Definition 4

For TDS (1), an LKF V is said to be

1. an ISS LKF with LKF-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \tag{4}$$

2. an ISS LKF with point-wise dissipation if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{5}$$

D^+V is the Driver's derivative of the functional V along the solution of (1).

General ISS characterization

Iasson Karafyllis, Pierdomenico Pepe, and Zhong-Ping Jiang (2008). "Input-to-output stability for systems described by retarded functional differential equations". In: *European Journal of Control* 14.6, pp. 539–555

Theorem 5

TDS (1) *is ISS if and only if it admits an ISS LKF with LKF-wise dissipation.*

Is the point-wise dissipation enough to ensure ISS of TDS?

Chaillet, Pepe, Mason, and Chitour 2017

Conjecture 1

Assume that the system (1) *admits a LKF with a point-wise dissipation. Then it is ISS.*

In response,

1. Antoine Chaillet, Pierdomenico Pepe, Paolo Mason, and Yacine Chitour (2017). "Is a point-wise dissipation rate enough to show ISS for time-delay systems?" In: *IFAC-PapersOnLine* 50.1, pp. 14356–14361
2. Antoine Chaillet, Iasson Karafyllis, Pierdomenico Pepe, and Yuan Wang (2023). "Growth conditions for global exponential stability and exp-ISS of time-delay systems under point-wise dissipation". In: *Systems & Control Letters* 178, p. 105570

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A way to get LKF-wise dissipation

Consider the LKF

$$V(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} V_2(x_t(s))ds, \tag{6}$$

which dissipates point-wisely as follow

$$D^+V \leq -\alpha(|x(t)|) + \gamma(|u(t)|).$$

By adding ke^{cs} in the kernel of integral part of (6), we get the LKF

$$W(x_t) = V_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} V_2(x_t(s))ds.$$

Proposition 1

If $\alpha(|x(t)|) \geq pV_2(x(t))$, $(p > 0)$ then W is a LKF with LKF-wise dissipation and the TDS (1) is ISS.

Does this trick work systematically?

Consider the following **1D** TDS

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{7}$$

and the LKFs:

$$V(x_t) := \frac{1}{4}x(t)^4 + \int_{-1}^{0} x_t(s)^4 ds,$$
$$W(x_t) := \frac{1}{4}x(t)^4 + \int_{-\Delta}^{0} ke^{cs} x_t(s)^4 ds.$$

Proposition 2

1. *System* (7) ***is ISS***.
2. V *is an ISS LKF with* ***point-wise dissipation*** *for* (7)
3. *Given any* $k, c > 0$, W *is not a LKF with LKF-wise dissipation.*

Sketch of the proof

Here $\Delta = 1$, we set $\mathcal{X} := C^0([-1,0],\mathbb{R})$ and we denote by f the dynamic of (7).

$$W \text{ dissipates LKF-wise} \Leftrightarrow D^+W(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|), \quad \forall \phi \in \mathcal{X}, v \in \mathbb{R}.$$

In particular for $v = 0$, this implies

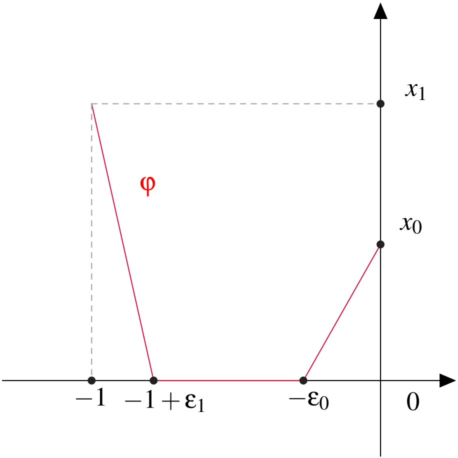
$$W \text{ dissipates LKF-wise} \Rightarrow D_0^+W := D^+W(\phi, f(\phi, 0)) \leq -\alpha(V(\phi)) \leq 0, \quad \forall \phi \in \mathcal{X}.$$

Sketch of the proof

Set $x_0 := \phi(0)$, $x_1 := \phi(-1)$.

$$D_0^+ W = x_0^4 \left(k - 1 - \frac{1}{1+x_0^2} \right) + x_1^4 \left(\frac{x_0^3}{1+|x_0|^3} - ke^{-c} \right) - kc \int_{-1}^{0} e^{cs} \phi(s)^4 ds.$$

$D^+ W(\varphi, f(\varphi, 0)) > 0.$



ISS under point-wise dissipation

Theorem 6

Assume that there exists a LKF V which dissipates point-wisely as follow,

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|). \tag{8}$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{9}$$

Then, if

$$\liminf_{r \to +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{10}$$

the system (1) *is ISS.*

Corollary 7

Assume that there exists a LKF V which dissipates point-wisely as follow,

$$D^+V \leq -Q(x(t)) + \gamma(|u(t)|). \tag{11}$$

Assume that

$$\dot{Q}(x(t)) \leq \sigma(\|Q\|) + \gamma(|u(t)|). \tag{12}$$

Then, if

$$\liminf_{r \to +\infty} \frac{r}{\sigma(r)} > 0, \tag{13}$$

the system (1) *is ISS.*

Conclusions of Part I

1. The principle of adding exponential term in relaxed LKFs to make them strict, does not systematically work even for $1D$ TDS.
2. Growth condition is proposed to conclude ISS with relaxed LKF.
3. The proposed condition turns out to extend the existing ones.
4. The conjecture is still an open question.

PART II

Rapid stabilization of general linear systems with F-equivalence

Joint work with Amaury Hayat

We consider the following linear PDE system

$$\partial_t u = \mathcal{A}u + Bw(t), \quad t \geq 0. \tag{14}$$

where $\mathcal{A}$ and B are linear operator, u the state of the system and w the control variable.

Objective

Provide conditions to stabilize the system (14), i.e

- Exhibit a feedback law K such that $w = Ku(t,\cdot)$ and the closed-loop system

$$\partial_t u = \mathcal{A}u + BKu, \quad t \geq 0 \tag{15}$$

  is well-posed;
- And show that any solution of (15) converges exponentially quick to 0, i.e

$$\|u(t,\cdot)\| \leq ke^{-\lambda t}\|u(0,\cdot)\|, \quad \forall t \geq 0. \tag{16}$$

Description of the used method

Let $\lambda > 0$. Assume that $\mathcal{A}$ generates a **dissipative** C^0 **semi-group** and consider the following target system

$$\partial_t v = (\mathcal{A} - \lambda I)v, \quad t \geq 0. \tag{17}$$

There exists $\omega > 0$ such that the system (17) converges exponentially to 0, i.e

$$\|v(t,\cdot)\| \leq k e^{-(\lambda+\omega)t} \|v(0,\cdot)\|, \quad \forall t \geq 0. \tag{18}$$

Assume that there exists (T, K) such that $v = Tu$ and (15) is well-defined.

Remark 1

If T is an isomorphism, we obtain

$$\|u(t,\cdot)\| \leq K e^{-(\lambda+\omega)t} \|u(0,\cdot)\|, \quad \forall t \geq 0, \tag{19}$$

where $K := k\|T^{-1}\|\|T\|$.

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Description of the used method

Recall that $\dot{u} = (\mathcal{A} + BK)u$ we have

$$v = Tu \quad \Leftrightarrow \quad \dot{v} = T\dot{u} = T(\mathcal{A} + BK)u.$$

Since $\dot{v} = (\mathcal{A} - \lambda I)v$, we get

$$T(\mathcal{A} + BK)u = (\mathcal{A} - \lambda I)Tu.$$

Goal

Provide isomorphism T and feedback K such that the following operator equality holds

$$T(\mathcal{A} + BK) = (\mathcal{A} - \lambda I)T. \tag{20}$$

There is no uniqueness of solutions of (20).

Uniqueness condition

To make the solutions of the operator equality unique, we introduce the condition :

$$TB = B.$$

The goal is now to solve the system:

$$T\mathcal{A} + BK = (\mathcal{A} - \lambda I)T \tag{21a}$$

$$TB = B. \tag{21b}$$

Do there always exist unique isomorphism T and feedback K solution of (21)?
What are the conditions on $\mathcal{A}$ and B if yes?

Answer in finite dimension

In finite dimension, i.e if $\mathcal{A}$ and B are matrices and $u \in \mathbb{R}^n$, the answer is yes.

Theorem 8 (Coron 2015)

Assume that $(\mathcal{A}, B)$ is controllable. There exists one and only one $(T, K) \in GL_n(\mathbb{R}) \times \mathbb{R}^n$ satisfying (21).

Recall the following

Theorem 9 (Coron 2007)

$(\mathcal{A}, B)$ is controllable if and only if

$$\textbf{\textit{rank}}\{B, AB, \cdots, A^i B, \cdots, A^{n-1} B\} = n.$$

Answer in infinite dimension?

1. Several results in the literature investigated the question
   - Gagnon, Hayat, Xiang, and Zhang 2022b where $\mathscr{A}$ is self-adjoint
   - Gagnon, Hayat, Xiang, and Zhang 2022a when $\mathscr{A}$ is a skew-adjoint
2. Can we extend these results to the framework of any "**spectral**" linear operator $\mathscr{A}$?

Spectral assumptions on $\mathcal{A}$

Assume that

1. $\mathcal{A}$ generates a dissipative C^0 semi-group on a Hilbert space X.
2. The eigenvectors φ_n of $\mathcal{A}$ form a Riesz basis of X.
3. The eigenvalues λ_n have finite multiplicity and there exists $\alpha > 1$ such that

$$n^\alpha \lesssim |\lambda_n| + 1 \lesssim n^\alpha, \quad \forall n \in \mathbb{N}^*, \tag{22}$$

$$|\lambda_n - \lambda_p| \gtrsim n^{\alpha-1}|n-p|, \quad \forall n,p \in \mathbb{N}^*. \tag{23}$$

Main result

Theorem 10

Let $\gamma \in [0, (\alpha - 1)/2)$. *If* B *satisfies*

$$c_1 \leq |\langle B, \widetilde{\varphi_n} \rangle| \leq c_2 n^\gamma, \quad \forall n \in \mathbb{N}^*, \tag{24}$$

Then, there exist a feedback K *and an isomorphism* T *such that* T *maps the system*

$$\partial_t u = \mathcal{A}u + BKu \tag{25}$$

to the system

$$\partial_t v = \mathcal{A}v - \lambda v. \tag{26}$$

For any $\mu > 0$, (25) *is exponentially stable with decay rate* $\mu > 0$.

$(\widetilde{\varphi_n})_n$ is the orthogonal basis of $(\varphi_n)_n$ as $\langle \varphi_n, \widetilde{\varphi_m} \rangle = \delta_{nm}$.

Contributions

Consider the following Bargmann space

$$X := \left\{ f : \mathbb{C} \to \mathbb{C} \text{ holomorphic } \mid \int_{\mathbb{C}} e^{-|z|^2} |f(z)|^2 dz < \infty \text{ and } f(0) = 0 \right\},$$

Let us consider the operators U and V defined as:

$$U : f \mapsto Uf = \frac{df}{dz} \quad \text{and} \quad V : f \mapsto Vf = zf$$

The Gribov operator is

$$\mathcal{A} := (VU)^3 + \varepsilon V(U+V)U + \varepsilon^2 (VU)^{3d_2} + \cdots + \varepsilon^k (VU)^{3d_k} + \cdots ,$$

This operator is neither self-adjoint nor skew-adjoint.

Contributions

Consider

$$\partial_t u = -\mathcal{A}u + \phi w(t) \tag{27}$$

Theorem 11

Let $\gamma \in [0,1)$. *If* ϕ *is such that there exists* $c_1, c_2 > 0$ *satisfying*

$$c_1 \leq \left| \int_{\mathbb{C}} e^{-|z|^2} \phi(z) \widetilde{\varphi_n}(z) dz \right| \leq c_2 n^{\gamma}, \quad \forall n \in \mathbb{N}^*.$$

Then there exists a feedback law $w(t) = K(u(t,\cdot))$ *such that the system* (27) *is exponentially stable in the Bargman space* X.

Contributions

We consider the bilinear Schroedinger equation linearized around the ground state.

$$\begin{aligned} &i\partial_t \Psi = -\Delta\Psi - \sigma_1 \Psi + u(t)\mu(x)\Phi_1(x), \\ &\Psi(t,0) = \Psi(t,1) = 0, \end{aligned} \tag{28}$$

where $\sigma_1 = \pi^2$ and $\Phi_1 = \sqrt{2}\sin(\pi x)$ and $\mu \in H^3(0,1)$.

Theorem 12

Assume that there exist constants $c, C > 0$ and $\gamma \in [0, 1/2)$ such that for all $n \in \mathbb{N}^*$

$$c \leq |\langle \mu\Phi_1, \varphi_n \rangle| \leq Cn^\gamma. \tag{29}$$

Then there exists K such that the system (28) *with $u(t,\cdot) = K(\Psi(t,\cdot))$ is exponentially stable in $H^3(0,1)$.*

Conclusions of Part II

1. We provide more relaxed condition to stabilize infinite dimensional system which involves a general spectral operator
2. The obtained conditions are weak controllability requirement and making our result strong
3. More investigations are needed to handle the case of infinite multiplicity of eigenvalues.

1. Part I

   Epiphane Loko, Antoine Chaillet, Yuan Wang, Iasson Karafyllis, and Pierdomenico Pepe (2024). “Growth conditions to ensure input-to-state stability of time-delay systems under point-wise dissipation”. In: *63rd IEEE Conference on Decision and Control (CDC 2024)*. IEEE

2. Part II

   Amaury Hayat and Epiphane Loko (2024). “Fredholm backstepping and rapid stabilization of general linear systems”. In: *Under review*

Thank you for your attention