Master Thesis

MASTER 2 MATHEMATICS, ANALYSIS AND APPLICATIONS

Novel point-wise dissipation conditions for input-to-state stability of time-delay systems

Written by

Epiphane LOKO

Supervised by

Prof Antoine CHAILLET

Prof Amaury HAYAT

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CONTENTS

INTRODUCTION

Many scientific questions are nowadays based on the uniformity of results between finite and infinite dimension. Several results which are true in finite dimension remain a puzzle in infinite dimension. This is the case of the Input-to-State Stability (ISS). It is today one of the powerful tools to study stability and robustness of nonlinear systems. Originally developped for systems described by ordinary differential equations [Sontag, 1989], it has progressively been extended to infinite-dimensional systems [Mironchenko and Prieur, 2020]. This notion of stability essentially imposes that the norm of any solution is bounded by a decaying term of the initial state's norm plus a term involving the amplitude of the applied input. Moreover, solutions of ISS systems converge to the origin in response to any vanishing input. More generally, they asymptotically converge to a neighborhood of the origin whose size is proportional to the magnitude of the applied input.
A particular class of infinite-dimensional sytems is that of time-delay systems. Due to the peculiarities of this subclass, it has been the subject of specific ISS developments initiated with the works [Teel, 1998] and [Pepe and Jiang, 2006]. ISS for time-delay systems has now become a mature topic, but some fundamental questions remain open.
A powerful tool to study ISS is the Lyapunov approach. It was shown in [Sontag and Wang, 1995] that ISS is equivalent to the existence of a Lyapunov function candidate whose derivative along the system's solutions is upper bounded by a negative unbounded dissipation rate involving the state norm plus a continuous term involving the input norm. For time-delay systems, the key difference is that the object of study is no longer a function, but rather a functional (as its argument is the whole state history over a bounded time interval). In [Karafyllis et al., 2008] and [Kankanamalage et al., 2017], the Lyapunov-Krasovskii functional characterization proposed for ISS requests that the dissipation of the LKF along the system's solutions is expressed in terms of the LKF itself (LKF-wise dissipation). This requirement turns out to be rather unhandy in practice. More crucially, it is not in line with the LKF characterization of global asymptotic stability, in which the LKF is allowed to dissipate merely in terms of the current value of the solution's norm (point-wise dissipation). To date, it is not known wheter ISS can be ensured through a point-wise dissipation. Bur, growth conditions under which a point-wise dissipation is enough to conclude ISS, are recently provided in [Chaillet et al., 2023a].
It is worth stressing that even in the absence of exogenous inputs, point-wise dissipation is not yet fully understood. In particular, the possibility to ensure global exponential stability with point-wise dissipation still constitutes an open question. A partial answer was given in [Chaillet et al., 2019, Chaillet et al. 2022], but only for systems ruled by a globally Lypschitz vector field. Two growth conditions were provided in [Chaillet et al., 2023a] to extends this result to a wider class of systems.
In this report, we significantly enlarge the class of system for which ISS can be estabilished under a point-wise dissipation. Our proofs being constructive, we actually provide several technical lemmas that may facilitate the understanding. To that end, we start by recalling some definitions and basic notions, then we recall all the main existing results on global asymptotic stability, global exponential stability and the ISS of delay systems. We finish by presenting our own investigations carried out during our internship and discuss the interest of what we have done, compared to what already exists.

Basic concepts and reminders

In this first part, we recall some basic notions and define the concepts of global asymptotic stability and global exponential stability. We start by introducing the notation that will be used throughout the document.

1.1 Notations

In this paper, $\mathbb{R}$ stands for the set of real number and $\mathbb{N}$ stands for the set of non-negative integers. Given $a \in \mathbb{R}$, $\mathbb{R}_{\geq a} := \{x \in \mathbb{R} : x \geq a\}$ and similary for $\mathbb{N}_{\geq a}$. Given $\Delta \geq 0$, and $n \in \mathbb{N}_{\geq 1}$, $\mathcal{X}^n$ denotes the space of the continuous functions mapping the interval $[-\Delta, 0]$ into $\mathbb{R}^n$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. The symbol ^{T} denotes the transpose of a matrix.
Given a non-empty (possibly unbounded) interval $\mathcal{I} \subset \mathbb{R}$ and a Lebesgue measurable signal $u : \mathcal{I} \to \mathbb{R}^m$, $\sup_{t\in\mathcal{I}} |u(t)| \in [0, +\infty]$ denotes the essential supremum of u, that is

$$\sup_{t\in\mathcal{I}} |u(t)| := \inf\{\bar{u} \geq 0 : \lambda(\{t \in \mathcal{I} : |u(t)| > \bar{u}\}) = 0\},$$

where λ denotes the Lebesgue measure. We also let $\|u\| := \sup_{t\in\mathcal{I}} |u(t)|$. The symbol $\mathcal{U}$ denotes the set of the Lebesgue measurable and locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Given $m \in \mathbb{N}_{\geq 1}$ and $u \in \mathcal{U}^m$, $u_{\mathcal{I}}$ denotes the restriction of u to $\mathcal{I}$, namely $u_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^m$ is defined as $u_{\mathcal{I}}(t) := u(t)$ for all $t \in \mathcal{I}$.

1.2 Comparison functions

The following class of functions are very useful to define and analyze stability of nonlinear systems.

Definition 1. *[Khalil, 1996, Definitions 4.2 and 4.3]*

- *A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{PD}$ if it satifies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$.*
- *A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if $\alpha \in \mathcal{PD}$ and it is increasing.*
- *$\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$.*
- *A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if it is non-increasing and tends to zero as its argument tends to infinity.*
- *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class $\mathcal{KL}$ function if $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \geq 0$, and $\beta(s, \cdot) \in \mathcal{L}$ for any fixed $s \geq 0$.*

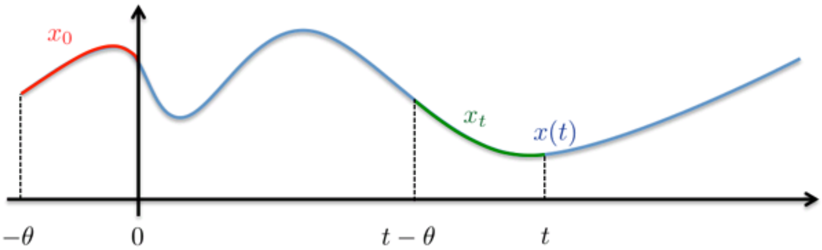

Figure 1.1: Time evolution of the solution of (1.1). Figure borrowed from [Chaillet et al., 2023b].

1.3 Basic concepts

1.3.1 Nonlinear time-delay system

The following definition presents the class of time-delay systems that we address in this document.

Definition 2. *A* ***Nonlinear time-delay system*** *is a system modeled by functional differential equation of the type:*

$$\dot{x}(t) = f(x_t, u(t)), \tag{1.1}$$

where $t \geq 0$ is the time variable, f is a map from $\mathcal{X}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ is the internal variable, $\dot{x}(t)$ is the right-hand derivative of $x(t)$ with respect to t, $u(t) \in \mathbb{R}^m$ is the input, and $x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the solution's history defined by $x_t(s) = x(t+s)$ for all $s \in [-\Delta, 0]$, where $\Delta \geq 0$ denotes the maximum time delay involved.

The state space of the system (1.1) is $\mathcal{X}^n$: the status of the system at some time $t \geq 0$ is not captured by current value of the solution $x(t) \in \mathbb{R}^n$ only, but rather from the history segment $x_t \in \mathcal{X}^n$ as depicted by Figure 1.1. This constitutes the main difference with respect to finite-dimensional systems: the state is no longer a point in $\mathbb{R}^n$, but rather a function in $\mathcal{X}^n$, thus explaining the infinite dimension of (1.1).

When we consider $u \equiv 0$, the system (1.1) becomes the following input-free system

$$\dot{x}(t) = f_0(x_t) \tag{1.2}$$

where $f_0(\phi) = f(\phi, 0)$ for all $\phi \in \mathcal{X}^n$.

1.3.2 Existence and uniqueness of solution

Before defining any notion of stability, it is necessary to ensure the existence of solutions. In this section, we define the solution of system (1.1) and then present some conditions for its existence and uniqueness.
For the system (1.1), the notion of solution is defined as follows.

Definition 3. *[Hale and Lunel, 1993] (Solution) Given an initial state $x_0 \in \mathcal{X}^n$ and an input signal $u \in \mathcal{U}^m$, a solution of* (1.1) *denotes any function $x : [0, t_{\max}) \to \mathbb{R}^n$, with* $t_{\max} \in (0, +\infty]$, *which is locally absolutely continuous and satisfies* (1.1) *almost everywhere in* $[0, t_{\max})$ *or, equivalently, satisfies the integral equation*

$$x(t) = x_0(0) + \int_0^t f(x_\tau, u(\tau))d\tau, \quad \forall t \in [0, t_{\max}).$$

In this document, we write $x(t, x_0, u) \in \mathbb{R}^n$ the maximal solution of (1.1) at time t corresponding to initial state $x_0 \in \mathcal{X}^n$ and input $u \in \mathcal{U}^m$. Similarly, $x_t(x_0, u) \in \mathcal{X}^n$ denotes the corresponding history segment, expressed in $\mathcal{X}^n$.

The following theorem ensures the existence and uniqueness of solution of (1.1). See [Hale and Lunel, 1993, Theorem 2.1, 2.2, 2.3, 3.1, 3.2], [Kolmanovskii and Myshkis, 1999, Theorem 2.1, 2.2]

Theorem 1. *Assume that the vector field f involved in* (1.1) *is Lipschitz on bounded sets and satisfies $f(0, 0) = 0$. The following results hold:*

- *for any initial state $x_0 \in \mathcal{X}^n$ and any input $u \in \mathcal{U}^m$,* (1.1) *admits a unique locally absolutely continuous solution $x(\cdot, x_0, u)$ on a maximal time interval* $[0, t_{\max})$ *with* $t_{\max} \in (0, +\infty]$
- *if* $t_{\max} < +\infty$, *then the solution is unbounded in* $[0, t_{\max})$

In what follows, we assume that the vector fields f_0 and f are Lipschitz on bounded sets and satisfy $f_0(0) = 0,\ f(0, 0) = 0$.

1.3.3 Forward completeness

While Theorem 1 ensures existence of solutions over some time interval $[0, t_{\max})$, it does not guarantee that $t_{\max} = +\infty$, as imposed by the following property.

Definition 4. *[Chaillet et al., 2023b, Definition 3] The system* (1.1) *is said to be forward complete (FC) if, for any initial state $x_0 \in \mathcal{X}^n$ and any input $u \in \mathcal{U}^m$, the corresponding solution exists on $\mathbb{R}_{\geq 0}$, i.e* $t_{\max} = +\infty$.

This property can actually be strengthened by additionaly imposing some uniformity on the set that can be reached by the solutions over a finite time interval, as imposed by the following property.

Definition 5. *[Karafyllis and Jiang, 2011, Definition 2.1] The system* (1.1) *is said to be robustly forward complete (RFC) if it is FC and , for every $T, r > 0$,*

$$\sup\{\|x_t(\phi, u)\| : \ \|\phi\| \leq r,\ t \in [0, T],\ u \in \mathcal{U}^m,\ \|u\| \leq r\} < +\infty.$$

In the other words, the RFC property means that, starting from any bounded set of initial states and considering inputs in any given bounded set, solutions can only evolve in a bounded region over any finite time interval.

1.3.4 Definition of global asymptotic and global exponential stability

Here, we recall the notions of global asymptotic stability and global exponential stability for the system (1.2). Before, we need to define the notion of equilibrium.

Definition 6. *A constant history function $\phi^* \in \mathcal{X}^n$ is said to be an equilibrium for the system* (1.2) *if and only it holds that*

$$f_0(\phi^*) = 0$$

In all that follows, we assume without loss of generality that $\phi^* = 0$ is the equilibrium point. Thus, we have the following definition of asymptotic and exponential stability.

Definition 7. *[Chaillet et al., 2023b, Definition 6] The origin of the system* (1.2) *is said to be:*

- *asymptotically stable (AS) if there exist $r > 0$ and $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathcal{X}^n$ with $\|x_0\| \leq r$, the corresponding solution of* (1.2) *satisfies*

$$|x(t, x_0)| \leq \beta(\|x_0\|, t), \quad \forall t \geq 0 \tag{1.3}$$

- *exponentially stable (ES) if there exist $r, \lambda > 0$ and $k \geq 1$ such that, for all $x_0 \in \mathcal{X}^n$ with $\|x_0\| \leq r$, the corresponding solution of* (1.2) *satisfies*

$$|x(t, x_0)| \leq k\|x_0\|e^{-\lambda t}, \quad \forall t \geq 0 \tag{1.4}$$

- *globally asymptotically stable (GAS) if there exists $\beta \in \mathcal{KL}$ such that* (1.3) *holds for all $x_0 \in \mathcal{X}^n$*

- *globally exponentially stable (GES) if there exist $\lambda > 0$ and $k \geq 1$ such that* (1.4) *holds for all $x_0 \in \mathcal{X}^n$.*

Remark 1. *It clear holds that GES $\Rightarrow$ GAS. As the function β defined by $\beta(t, s) = kte^{-\lambda s}$ is clearly of class $\mathcal{KL}$.*

1.3.5 Lyapunov-Krasovskii functional

A powerful tool to study the stability of nonlinear time-delay systems is the Lyapunov-Krasovskii approch. This method require to find a proper functional that decreases along the solutions of the sytem. We start by defining the class of functionals considered for that purpose.

Definition 8. *[Karafyllis and Jiang, 2011, Krasovskii 1963] A functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov-Krasovskii functional candidate (LKF) if it is Lipschitz on bounded sets and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{1.5}$$

It is said to be a coercive LKF if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{1.6}$$

Note that in inequality (1.5), the upper and lower bounds on LKF V differ: the upper bound involves whole norm $\|\phi\|$ of the state history whereas the lower bound involves solely the current value of the solution's norm $|\phi(0)|$. Condition (1.5) imposes that $V(0) = 0$, that V is positive whenever $\phi(0) \neq 0$, and that it grows to infinity as $|\phi(0)| \to +\infty$. Condition (1.6) is a stronger requirement as it imposes additionally that V does not vanish unless ϕ is identically zero.

1.3.6 Functional derivatives

Since for time delay systems, the object of study is no longer a function but rather a functional, we need to define a particular notion of functional differentiation. In this document, we mostly make use of Driver's derivative, as recalled next.

Definition 9. *[Driver, 1962] (Driver's derivative) Let* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *be a continuous functional. Its Driver's derivative* $D^+V : \mathcal{X}^n \times \mathbb{R}^n \to \bar{\mathbb{R}}$ *is defined, for all* $\phi \in \mathcal{X}^n$ *and all* $w \in \mathbb{R}^n$, *as*

$$D^+V(\phi, w) := \limsup_{h \to 0^+} \frac{V(\phi_{h,w}) - V(\phi)}{h}$$

where for each $h \in [0, \Delta), \phi_{h,w} \in \mathcal{X}^n$ *is given by*

$$\phi_{h,w}(s) := \begin{cases} \phi(s+h), & \text{if } s \in [-\Delta, -h), \\ \phi(0) + w(h+s), & \text{if } s \in [-h, 0) \end{cases}$$

In practice, we often want to study Driver's derivative in the direction imposed by the vector field of (1.1), in which case we simply let $w = f(\phi, v)$, with $\phi \in \mathcal{X}^n$ and $v \in \mathbb{R}^m$, in the above definition.
The literature on time-delay systems often uses Dini's derivative, as recalled next.

Definition 10. *Let* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *be a continuous functional. Its upper-right Dini's derivative* $\mathcal{D}^+V : \mathcal{X}^n \to \overline{\mathbb{R}}$ *along the solution of* (1.1) *is defined by,*

$$\mathcal{D}^+V(x_t) := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}.$$

It is worth stressing that these two derivatives are tightly linked, in the sense that $D^+V(x_t, f(x_t, u(t))) = \mathcal{D}^+V(x_t)$ almost everywhere on the interval of existence of the solution x_t.

Existing results

2.1 Global asymptotic stability (GAS)

In finite dimension, a system is GAS if and only if its origin is both stable and globally attractive. To date, it is not known whether the same holds for time-delay systems (see the discussion in [Karafyllis et al., 2022]). Nevertheless, it has recently been shown that this equivalence holds under the RFC condition (recalled in Definition 5), as recalled next.

Theorem 2. *Assume that system* (1.2) *is RFC. Then it is GAS if and only if the following two properties hold:*

- *(stability) for each $\varepsilon > 0$ there exist $\delta > 0$ such that, for any $x_0 \in \mathcal{X}^n$ with $\|x_0\| \leq \delta$, the corresponding solution of* (1.2) *satisfies $|x(t, x_0)| \leq \varepsilon$ for all $t \geq 0$*
- *(global attractivity) for any $x_0 \in \mathcal{X}^n$, the corresponding solution of* (1.2) *satisfies $\lim_{t\to+\infty} x(t, x_0) = 0$.*

The following result shows that asymptotic stability can be characterized in LKF terms. See [Karafyllis and Jiang, 2011, Theorem 30.2, 31.3], [Pepe and Karafyllis, 2013, Theorem 2.3], and [Chaillet et al., 2022, Proposition 2].

Theorem 3. *The following statements are equivalent:*

- (1.2) *is GAS*
- *there exist an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ and $\sigma \in \mathcal{KL}$ such that, for all $\phi \in \mathcal{X}^n$,*

$$D^+V(\phi, f_0(\phi)) \leq -\sigma(|\phi(0)|, ||\phi||) \tag{2.1}$$

- *there exist an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ and $\alpha \in \mathcal{P}$ such that, for all $\phi \in \mathcal{X}^n$,*

$$D^+V(\phi, f_0(\phi)) \leq -\alpha(|\phi(0)|) \tag{2.2}$$

- *there exist an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ such that, for all $\phi \in \mathcal{X}^n$,*

$$D^+V(\phi, f_0(\phi)) \leq -V(\phi) \tag{2.3}$$

- *there exist a coercive LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ and $\alpha \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,*

$$D^+V(\phi, f_0(\phi)) \leq -\alpha(||\phi||) \tag{2.4}$$

Condition (2.2) is probably the handiest way to estabilish GAS as the considered LKF is not requested to be coercive and its dissipation rate is only required to involved the current value of the solution's norm (point-wise dissipation). It can actually be even relaxed to (2.1), in which the dissipation rate is allowed to be smaller when $||\phi||$ gets larger ($\mathcal{KL}$ dissipation). It is interesting to notice that any of these two conditions ensures the existence of an LKF which dissipates exponentially along the systems solution (condition (2.3)), and even the existence of a coercive LKF that dissipates in terms of the whole state history (history-wise dissipation: condition (2.4)), which may prove useful for futher robustness analysis.
These equivalences are expected for GES and ISS. But we will see in the following that this is not always the case, or at least that we are not yet sure.

2.2 Global Exponential Stability (GES)

The following result shows that GES can be characterized through an LKF, provided that we can get a LKF-wise or a history-wise dissipation. See [Karafyllis and Jiang, 2011, Lemma 33.1],[Pepe and Karafyllis, 2013, Theorem 2.3, 2.4, 2.5], [Haidar et al., 2022].

Theorem 4. *The following statements are equivalent:*

- (1.2) *is GES*
- *there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a, p > 0$ *such that, for all* $\phi \in \mathcal{X}^n$,

$$\underline{a}|\phi(0)|^p \leq V(\phi) \leq \overline{a}||\phi||^p \tag{2.5}$$
$$D^+V(\phi, f_0(\phi)) \leq -aV(\phi) \tag{2.6}$$

- *there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a, p > 0$ *such that, for all* $\phi \in \mathcal{X}^n$,

$$\underline{a}||\phi||^p \leq V(\phi) \leq \overline{a}||\phi||^p$$
$$D^+V(\phi, f_0(\phi)) \leq -a||\phi||^p$$

Moreover, the same equivalence holds for ES if the above bound hold only for $||\phi|| \leq r$ *for some* $r > 0$.

To date it is not known yet wheter GES can be estabilished through a point-wise dissipation as in the GAS case. Nevertheless, additional growth conditions have been proposed recently to make this possible.

Theorem 5. *[Chaillet et al., 2019, Theorem 1] Assume that* f_0 *is globally Lipschitz. The following statements are equivalent:*

- (1.2) *is GES*
- *there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a > 0$ *such that, for all* $\phi \in \mathcal{X}^n$,

$$\underline{a}|\phi(0)|^2 \leq V(\phi) \leq \overline{a}\|\phi\|^2$$
$$D^+V(\phi, f_0(\phi)) \leq -aV(\phi)$$

- *there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a > 0$ *such that, for all* $\phi \in \mathcal{X}^n$,
$$\begin{aligned} \underline{a}\|\phi\|^2 \leq V(\phi) \leq \overline{a}\|\phi\|^2 \\ D^+V(\phi, f_0(\phi)) \leq -a\|\phi\|^2 \end{aligned}$$
- *there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a > 0$ *such that, for all* $\phi \in \mathcal{X}^n$,
$$\begin{aligned} \underline{a}|\phi(0)|^2 \leq V(\phi) \leq \overline{a}\|\phi\|^2 \\ D^+V(\phi, f_0(\phi)) \leq -a|\phi(0)|^2 \end{aligned}$$

This result is of interest as several real-world systems can be modeled with a globally Lipschitz vector field. But this global Lipschitz condition remains restrictive. The following result extends the class of systems beyond globally Lipschitz functions.

Theorem 6. *[Chaillet et al., 2023a, Corollary 1, 2] Assume that there exist a functional* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *which is Lipshitz on bounded sets,* $\underline{a}, \overline{a}, a > 0$, *and* $\varepsilon \geq 0$ *such that, for all* $\phi \in \mathcal{X}^n$,
$$\begin{aligned} \underline{a}|\phi(0)|^2 \leq V(\phi) \leq \overline{a}\|\phi\|^2 \\ D^+V(\phi, f_0(\phi)) \leq -a|\phi(0)|^2 + \varepsilon\|\phi\|^2. \end{aligned}$$
Assume further that there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ *and a constant* $c > 0$ *such that any of the two following conditions is satisfied:*
$$\phi(0)^T P f_0(\phi) \leq c\|\phi\|^2, \quad \forall \phi \in \mathcal{X}^n \tag{2.7}$$
$$\phi(0)^T P f_0(\phi) \geq -c\|\phi\|^2, \quad \forall \phi \in \mathcal{X}^n \tag{2.8}$$
Then there exists $\bar{\varepsilon}$ *such that, if* $\varepsilon \in [0, \bar{\varepsilon})$, *the system* (1.2) *is GES.*

Remark 2. *If* f_0 *is globally Lipschitz, there exist* P *such that conditions* (2.7) *and* (2.8) *hold. Hence, Theorem 6 is more general than Theorem 5. Some systems with vector fields that are not globally Lipschitz and for which GES can be estabilished with Theorem 6 were actually presented in [Chaillet et al., 2023a].*

Unlike Theorems 5 and 6, the following does not require any growth condition to guarantee exponential stability with point-wise dissipation. Nevertheless, it provides only local exponential stability.

Theorem 7. *[Chaillet et al., 2019] The origin of* (1.2) *is ES if and only if there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a, r > 0$ *such that, for all* $\phi \in \mathcal{X}^n$ *with* $\|\phi\| \leq r$,
$$\begin{aligned} \underline{a}|\phi(0)|^2 \leq V(\phi) \leq \overline{a}\|\phi\|^2 \\ D^+V(\phi, f_0(\phi)) \leq -a|\phi(0)|^2. \end{aligned}$$

As mentioned above, we do not know to date whether the last result is true or not in the global case, but this has been conjectured in [Chaillet et al., 2017, Conjecture 1].

Conjecture 1. *Let the map* f_0 *in* (1.2) *be Lipschitz on bounded sets with* $f_0(0) = 0$ *and assume that there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *and* $\underline{a}, \overline{a}, a > 0$ *such that for all* $\phi \in \mathcal{X}^n$,
$$\begin{aligned} \underline{a}|\phi(0)|^2 \leq V(\phi) \leq \overline{a}\|\phi\|^2 \\ D^+V(\phi, f_0(\phi)) \leq -a|\phi(0)|^2. \end{aligned}$$
Then the system (1.2) *is GES.*

This conjecture remains to this day an open question and will probably be the subject of our future investigations.

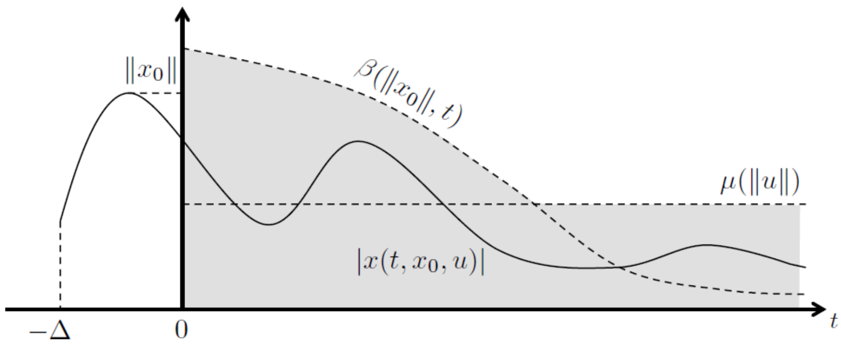

Figure 2.1: Schematic representation of the time evolution of an ISS system's solution. Borrowed from [Chaillet et al., 2023b].

2.3 Input-to-State Stability (ISS)

2.3.1 Definition and ISS properties

We are now going to talk about another form of stability, in particular for the system (1.1). The stability which we are going to define here is called *Input-to-State Stability (ISS)*. It allows to evaluate the impact of the input u on the system's behavior.

Definition 11. *[Teel, 1998] System* (1.1) *is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$,*

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{2.9}$$

The function μ is then called an ISS gain.

Remark 3. *As in the finite-dimensional case [Sontag, 1989], the estimate* (2.9) *expresses the asymptotically vanishing effect of the initial conditions (via the term $\beta(||x_0||, t)$) and the persistent effect of the external input u (via the term $\mu(||u_{[0,t]}||)$). This is illustrated by Figure 2.1.*
It is worth stressing that the ISS gain μ is here considered to be of class $\mathcal{K}_\infty$. The ISS literature often imposes that $\mu \in \mathcal{K}$ or even $\mu \in \mathcal{N}$ (i.e continuous, non-decreasing and zero at zero functions). The key property is that the ISS gain vanishes continuously at zero, which ensures that inputs of small magnitude produce small steady-state errors. Note also that any function of class $\mathcal{K}_\infty$ can be upper bounded by a $\mathcal{K}$ or $\mathcal{N}$ function, thus making the state estimate (2.9) *also satisfied with these classes of ISS gains.*

Futhermore, as in the finite-dimensional case, ISS automatically guarantees the following properties:

- 0-GAS: the input-free system $\dot{x}(t) = f(x_t, 0)$ is GAS

- *bounded input-bounded state* (BIBS) property: given any $x_0 \in \mathcal{X}^n$ and any $u \in \mathcal{U}^m$,

$$||u|| < +\infty \quad \Rightarrow \quad \sup_{t \geq 0} |x(t, x_0, u)| < +\infty \tag{2.10}$$

- *asymptotic gain* (AG) property: there exist $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$:
$$\limsup_{t\to+\infty} |x(t, x_0, u)| \leq \mu(||u||) \tag{2.11}$$

- *converging input-converging state* (CICS) property: given any $x_0 \in \mathcal{X}^n$ and any $u \in \mathcal{U}^m$,
$$\lim_{t\to+\infty} |u(t)| = 0 \quad \Rightarrow \quad \lim_{t\to+\infty} |x(t, x_0, u)| = 0 \tag{2.12}$$

Equivalent way to define ISS (given in [Chaillet et al., 2023b, page 21]) is to replace estimate (2.9) by

$$|x(t, x_0, u)| \leq \max\{\beta(||\phi||, t), \mu(||u_{[0,t]}||)\}, \quad \forall t \geq 0.$$

Figure 2.1 shows more this formulation.

2.3.2 Characterizations of ISS by LKF tools

As we have done in the GAS and GES case, we enumerate some existing results to characterize ISS by using Lyapunov approch. They rely on the following notions, which differ in the way the LKF dissipates along the system's solutions.

Definition 12. *[Chaillet et al., 2023b, Definition 12] (ISS LKF) For system* (1.1)*, an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *is said to be*

- *an ISS LKF with history-wise dissipation if there exist* $\alpha \in \mathcal{K}_\infty$ *and* $\gamma \in \mathcal{K}_\infty$ *such that*
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(||\phi||) + \gamma(|v|) \tag{2.13}$$

- *an ISS LKF with LKF-wise dissipation if there exist* $\alpha \in \mathcal{K}_\infty$ *and* $\gamma \in \mathcal{K}_\infty$ *such that*
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|) \tag{2.14}$$

- *an ISS LKF with point-wise dissipation if there exist* $\alpha \in \mathcal{K}_\infty$ *and* $\gamma \in \mathcal{K}_\infty$ *such that*
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|) \tag{2.15}$$

- *an ISS LKF in implication form if there exist* $\alpha \in \mathcal{P}$ *and* $\chi \in \mathcal{K}_\infty$ *such that*
$$V(\phi) \geq \chi(|v|) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) \tag{2.16}$$

where (2.13)-(2.16) *are all meant to hold for all* $\phi \in \mathcal{X}^n$ *and all* $v \in \mathbb{R}^m$.

Remark 4. *The following implications hold:*

$$(2.13) \Rightarrow (2.14) \Rightarrow (2.15)$$
$$(2.14) \Rightarrow (2.16)$$

We have then the following characterization (see [Karafyllis et al., 2008, Theorem 3.3] and [Kankanamalage et al., 2017, Theorem 2]).

Theorem 8. *(LKF characterizations of ISS) The following properties are equivalent:*

1. (1.1) *is ISS*
2. (1.1) *admits a coercive ISS LKF with history-wise dissipation*
3. (1.1) *admits an ISS LKF with LKF-wise dissipation*
4. (1.1) *admits an ISS LKF in implication form.*

Let us notice that all of the previous characterizations do not ensure ISS with an ISS LKF that dissipates point-wisely. In fact, this has been conjectured in [Chaillet et al., 2023b, Conjecture 4].

Conjecture 2. *Assume that there exist an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$; $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,*

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|).$$

Then the system (1.1) *is ISS.*

But this remains an open question. We will expose in the next chapter our different investigations in this direction. But first, let us consider the following result given in [Chaillet et al., 2017], which provides sufficient conditions to ensure ISS with an LKF that dissipates point-wisely.

Theorem 9. *(ISS through point-wise dissipation rate) Consider an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ for which there exist $\underline{\alpha}, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}_1(|\phi(0)|) + \overline{\alpha}_2\left(\int_{-\Delta}^{0} \overline{\alpha}_3(|\phi(\tau)|)d\tau\right). \tag{2.17}$$

Assume that V is an ISS LKF with point-wise dissipation for (1.1)*, meaning that it satisfies* (2.15) *for some $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$. Then, under the condition that*

$$\liminf_{s \to +\infty} \frac{\alpha(s)}{\overline{\alpha}_3(s)} > 0, \tag{2.18}$$

the system (1.1) *is ISS.*

ISS of systems (1.1) can also be demonstrated under the RFC requirement.

Theorem 10. *[Jacob et al., 2020, Theorem 4.4] Assume that* (1.1) *is RFC and that there exist a functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ which is Lipschitz on bounded sets of $\mathcal{X}^n$, $\alpha, \overline{\alpha} \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,*

$$\begin{gathered} 0 < V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \neq 0, \\ D^+V(\phi, f(\phi, v)) \leq -\alpha(\|\phi\|) + \gamma(|v|) \end{gathered}$$

Then the system (1.1) *is ISS.*

2.4 Exponential-ISS

In this section we will define exponential ISS, which is a particular case of ISS in which the influence of the initial state is requested to decay exponentially. In particular, exp-ISS ensures GES for the corresponding input-free system.

Definition 13. *[Chaillet et al., 2023a, Definition 2] The system* (1.1) *is said to be exponentially input-to-state stable (exp-ISS) if there exist $k, \eta > 0$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$, the corresponding solution of* (1.1) *satisfies*

$$|x(t, x_0)| \leq k\|x_0\|e^{-\eta t} + \mu(\|u_{[0,t]}\|), \quad \forall t \geq 0. \tag{2.19}$$

It is said to be exp-ISS with linear gain if, in addition, there exists $\mu_0 \geq 0$ such that $\mu(s) = \mu_0 s$ for all $s \geq 0$.

Remark 5. *As we can notice, the exp-ISS is a combination of $0-GES$ and ISS. It is actually a particular case of ISS, in which the decaying term is requested to be exponential. It can also be seen as a generalization of the exponential stability as ES is the particular case where $u \equiv 0$.*

Exp-ISS can be estabilished by existing LKF tools.

Proposition 1. *[Chaillet et al., 2023a, Theorem 2] Assume that there exist a functionnal $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ which is Lipschitz on bounded sets, constants $\underline{a}, \overline{a}, a, p > 0$ and a function $\gamma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,*

$$\begin{aligned} &\underline{a}|\phi(0)|^p \leq V(\phi) \leq \overline{a}\|\phi\|^p \\ &D^+V(\phi, f(\phi, v)) \leq -aV(\phi) + \varepsilon\|\phi\|^p + \gamma(|v|). \end{aligned}$$

Then, provided that

$$\varepsilon < \underline{a}ae^{-a\Delta}$$

the system (1.1) *is exp-ISS. Moreover, if there exists $\gamma_0 \geq 0$ such that $\gamma(s) = \gamma_0 s^p$ for all $s \geq 0$, then* (1.1) *is exp-ISS with linear gain.*

Although allowing a positive term involving the whole history segment's norm, the above result requests a LKF-wise dissipation.
In the following, we will present a result in which the exp-ISS is ensured with an LKF that disipates point-wisely by imposing specific conditions on the system dynamics. These conditions may take two forms: either left or right, depending on whether the increase rate or the decay rate is restricted.

Theorem 11. *[Chaillet et al., 2023a, Theorems 3 & 4] Assume that there exist a functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ which is Lipshitz on bounded sets, $, \overline{a}, a > 0$, $\varepsilon \geq 0$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and $v \in \mathbb{R}^n$,*

$$\begin{aligned} 0 \leq V(\phi) &\leq \overline{a}\|\phi\|^2 \\ D^+V(\phi, f_0(\phi)) &\leq -a|\phi(0)|^2 + \varepsilon\|\phi\|^2 + \gamma(|v|). \end{aligned}$$

Assume further that there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ *and a constant* $c > 0$ *such that any of the two following conditions is satisfied:*

$$\phi(0)^T P f_0(\phi) \leq c(\|\phi\|^2 + \gamma(|v|)), \quad \forall \phi \in \mathcal{X}^n \quad \textbf{\textit{(Right growth condition)}}$$
$$\phi(0)^T P f_0(\phi) \geq -c(\|\phi\|^2 + \gamma(|v|)), \quad \forall \phi \in \mathcal{X}^n \quad \textbf{\textit{(Left growth condition)}}$$

Then there exists $\bar{\varepsilon}$ *such that, if* $\varepsilon \in [0, \bar{\varepsilon})$, *the system* (1.1) *is exp-ISS. If, in addition, there exists* $\gamma_0 \geq 0$ *such that* $\gamma(s) = \gamma_0 s^2$ *for all* $s \geq 0$, *then* (1.1) *is exp-ISS with linear gain.*

Remark 6. *Note that Theorem 6 is a particular case of Theorem 11: it corresponds to the case of an input-free system (i.e* $u \equiv 0$*) and* $\varepsilon = 0$.

Results

Previously we have seen that the only results that ensure the ISS with a point-wise dissipating LKF are Theorems 9 and 11. And even if they do not answer exactly to Conjecture 2, they allow to consider a remarkable class of systems for which the conjecture is true. Here, we will further extend this class by proposing results that ensure the ISS under pointwise dissipation.

3.1 Statement of results

The following is the main result that we established.

Theorem 12. *Assume that there exist a functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ which is Lipschitz on bounded sets, $\bar{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ and $Q : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ continuously differentiable, positive definite and radially unbounded such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,*

$$0 \leq V(\phi) \leq \bar{\alpha}(\|\phi\|) \tag{3.1}$$

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|) \tag{3.2}$$

Assume further that there exists a function $\sigma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma\left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau))\right) + \gamma(|v|). \tag{3.3}$$

Then, under the condition that

$$\liminf_{r \to \infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{3.4}$$

the system (1.1) *is ISS.*

- A first remark is that the lower bound of inequality (3.1) is 0 and we did not need to impose the RFC condition as in Theorem 10.

- Condition (3.3) constitutes a mild requirement because in practice it is easy to construct σ that satisfies (3.3) when Q is known. The main requirement in the above statement is (3.4), which imposes that the term σ in the bound (3.3) is dominated at infinity by the dissipation rate α.

- If we consider Q such that $Q(x) = x^T P x$ with $P \in \mathbb{R}^{n \times n}$ symetric positive definite, and we impose on $\alpha, \bar{\alpha}$ and σ to be quadratic as well, we notice that Theorem 12 becomes exactly Theorem 11. Thus, Theorem 11 is a special case of Theorem 12.

- Professor Yuan Wang gave us an example (see Example 2, section 3.2) of a system for which Theorem 11 does not allow to prove ISS, but our previous Theorem 12 does. Hence, our Theorem 12 enlarges the class of systems for which conjecture 2 is true.

The following statement, although less general, provides an easier condition to check. It is a consequence of Theorem 12 when choosing α as the identity.

Corollary 1. *Assume that there exist a functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ which is Lipschitz on bounded sets, $\bar{\alpha}, \gamma \in \mathcal{K}_\infty$ and $Q : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ continuously differentiable, positive definite and radially unbounded such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,*

$$0 \leq V(\phi) \leq \bar{\alpha}(\|\phi\|) \tag{3.5}$$
$$D^+V(\phi, f(\phi, v)) \leq -Q(\phi(0)) + \gamma(|v|) \tag{3.6}$$

Assume further that there exists a function $\sigma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma\left(\max_{\tau\in[-\Delta,0]} Q(\phi(\tau))\right) + \gamma(|v|). \tag{3.7}$$

Then, under the condition that

$$\liminf_{r\to\infty} \frac{r}{\sigma(r)} > 0, \tag{3.8}$$

the system (1.1) *is ISS.*

The particularity of this result lies in the fact that the dissipation rate is not a class $\mathcal{K}_\infty$ function but rather a positive definite function defined on $\mathbb{R}^n$. This offers ISS with point-wise dissipation by using non necessarly increasing rate of dissipation and thus goes beyond the characterization proposed in the Conjecture 2. It should be noted that any continuously differentiable, positive definite and radially unbounded function is bounded by class $\mathcal{K}_\infty$ functions. Thus, we could always obtain the characterization of the Conjecture 2 from the one of the Corollary 1. This is precisely the aspect brought out by the following corollary.

Corollary 2. *Assume that there exist a functional $V : \mathcal{X}^n \to \mathbb{R}_+$ which is Lipschitz on bounded sets, $\bar{\alpha}, \gamma \in \mathcal{K}_\infty$, and a continuously differentiable $\alpha \in \mathcal{K}_\infty$ satisfying $\alpha'(0) = 0$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,*

$$0 \leq V(\phi) \leq \bar{\alpha}(\|\phi\|) \tag{3.9}$$
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|) \tag{3.10}$$

Assume further that there exist a function $\sigma \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma(\|\phi\|) + \gamma(|v|). \tag{3.11}$$

Then, under the condition that

$$\liminf_{r\to\infty} \frac{\alpha(r)}{\sigma(r)} > 0, \tag{3.12}$$

the system (1.1) *is ISS.*

The proofs are provided in section 3.3. They rely on the following lemmas.

Lemma 1. *[Karafyllis and Jiang, 2011, Lemma 6.7] Given $Q \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}_+)$ and $c > 0$, the functional $V : \mathcal{X}^n \to \mathbb{R}_+$ defined as*

$$V(\phi) := \max_{\tau\in[-\Delta,0]} e^{2c\tau} Q(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n,$$

is Lipschitz on bounded sets and satisfies, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V(\phi) > Q(\phi(0)) \Rightarrow D^+V(\phi, f(\phi, v)) \leq -2cV(\phi) \tag{3.13}$$

$$V(\phi) = Q(\phi(0)) \Rightarrow D^+V(\phi, f(\phi, v)) \leq \max\{-2cV(\phi), \nabla Q(\phi(0))f(\phi, v)\}. \tag{3.14}$$

Lemma 2. *Let $f, g \in \mathcal{K}_\infty$ be such that $\liminf\limits_{s\to\infty} \frac{f(s)}{g(s)} > 0$. Then, there exist a positive definite, non-decreasing and bounded function ω such that $\omega(s) \leq \frac{f(s)}{g(s)}$.*

The proof of this previous lemma 2 is given in Section 3.3.1.

Lemma 3. *[Ito et al., 2010, Lemma 7] Given any locally Lipschitz functional $V : \mathcal{X}^n \to \mathbb{R}_+$ and any continuous function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, let $W : \mathcal{X}^n \to \mathbb{R}_+$ be the continuous functional defined as*

$$W(\phi) := \int_0^{V(\phi)} \lambda(s)ds, \quad \forall \phi \in \mathcal{X}^n.$$

Then, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$, it holds that

$$D^+W(\phi, f(\phi, v)) \leq \lambda(V(\phi))D^+V(\phi, f(\phi, v)).$$

In many cases, when we have an LKF that dissipates point-wisely, it is often convenient to construct another LKF-wise dissipation by adding a exponential term in the kernel of the integral. Still, this trick is not guaranteed to work with any LKF. We give here a sufficient condition which allows to build this LKF.

Theorem 13. *Assume that W is an LKF such that for all $\phi \in \mathcal{X}^n$:*

$$W(\phi) = V_1(\phi(0)) + \int_{-\Delta}^0 V_2(\phi(s))ds, \tag{3.15}$$

$$D^+W(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|) \tag{3.16}$$

where $\alpha \in \mathcal{K}_\infty$ and $Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuously differentiable, positive definite and radially unbounded. If there exist $\varepsilon > 0$ such that for all $\phi \in \mathcal{X}^n$:

$$\alpha(Q(\phi(0))) \geq \varepsilon V_2(\phi(0)), \tag{3.17}$$

then there exist $c, p > 0,\ \lambda \in \mathcal{K}_\infty$ such that the LKF V defined by

$$V(\phi) = V_1(\phi(0)) + p\int_{-\Delta}^0 e^{cs}V_2(\phi(s))ds \quad \forall \phi \in \mathcal{X}^n \tag{3.18}$$

satisfies

$$D^+V(\phi, f(\phi, v)) \leq -c\lambda(V(\phi)) + \gamma(|v|) \quad \forall \phi \in \mathcal{X}^n \tag{3.19}$$

and then the system (1.1) is ISS.

The above result provides conditions under which, starting from a point-wise dissipation, one can construct an LKF with LKF-wise dissipation and thus conclude to ISS with existing tools. This result remains true when condition (3.16) is replaced by condition (3.10). Moreover, it should be noted that the conditions in this one are similar to those in Theorem 9. In particular if we consider $V_2(x) = \overline{\alpha}_3(|x|)$, then the condition (2.18) implies condition (3.17) at infinity. The following lemma is useful for its proof which is given in section 3.3.4.

Lemma 4. *[Kellett, 2014, Lemma 9] Let $\alpha \in \mathcal{K}$, and $a, b \geq 0$. Then*

$$\alpha(a+b) \leq \alpha(2a) + \alpha(2b).$$

3.2 Examples

We illustrate our main results through the following examples provided to us by the professor Yuan Wang. The following example, give a 2×2 system that is ISS by our Theorem 12.

Example 1. *Consider the system*

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + x_2(t-\Delta)^3 + u(t) \\ \\ \dot{x}_2(t) = -x_2(t) - x_2(t)^9 \end{cases} \tag{3.20}$$

and V defined by

$$V(\phi) = \frac{1}{4}\phi_1(0)^4 + \frac{1}{4}\phi_2(0)^4 + \int_{-\Delta}^{0} \phi_2(s)^{12} ds. \tag{3.21}$$

So, we have

$$\begin{aligned} D^+V(\phi, f(\phi, v)) &= -2\phi_1(0)^4 + \phi_1(0)^3\phi_2(-\Delta)^3 + \phi_1(0)^3 v - \phi_2(0)^4 - \phi_2(0)^{12} + \phi_2(0)^{12} - \phi_2(-\Delta)^{12} \\ &\leq -2\phi_1(0)^4 + \frac{3}{4}\phi_1(0)^4 + \frac{1}{4}\phi_2(-\Delta)^{12} + \frac{3}{4}\phi_1(0)^4 + \frac{v^4}{4} - \phi_2(0)^4 - \phi_2(-\Delta)^{12} \\ &\leq -\frac{1}{2}\phi_1(0)^4 - \frac{3}{4}\phi_2(-\Delta)^{12} - \phi_2(0)^4 + \frac{v^4}{4} \\ &\leq -\frac{1}{2}\left(\phi_1(0)^4 + \phi_2(0)^4\right) + \frac{v^4}{4} \\ &\leq -\frac{1}{4}|\phi(0)|^4 + \frac{v^4}{4}. \end{aligned} \tag{3.22}$$

We can take

$$\forall\ s \geq 0,\ \ \alpha(s) = \frac{1}{2}s^2$$
$$\forall\ x \in \mathbb{R}^2,\ \ Q(x) = \frac{1}{2}|x|^2$$

and then, (3.22) *is rewriten as*

$$D^+V(\phi) \leq -\alpha(Q(\phi(0))) + \frac{v^2}{4}$$

which ensure (3.2). *Moreover*

$$\begin{aligned}
\nabla Q(\phi(0))f(\phi,v) &= -2\phi_1(0)^2+\phi_1(0)\phi_2(-\Delta)^3+\phi_1(0)v-\phi_2(0)^2-\phi_2(0)^{10}\\
&\le -2\phi_1(0)^2+\frac{1}{4}\phi_1(0)^4+\frac{3}{4}\phi_2(-\Delta)^4+\frac{1}{2}(\phi_1(0)^2+v^2)-\phi_2(0)^2-\phi_2(0)^{10}\\
&\le -\frac{3}{2}\phi_1(0)^2-\phi_2(0)^2-\phi_2(0)^{10}+\frac{3}{4}(\phi_1(0)^4+\phi_2(-\Delta)^4)+\frac{v^2}{2}\\
&\le 3\left(\max_{\tau\in[-\delta,0]}Q(\phi(\tau))\right)^2+\frac{v^2}{2}\\
&\le \sigma_1\left(\max_{\tau\in[-\delta,0]}Q(\phi(\tau))\right)+\frac{v^2}{2}
\end{aligned}$$

where $\sigma_1(s)=3s^2$. *So,*

$$\liminf_{r+\infty}\frac{\alpha(r)}{\sigma_1(e^{2\Delta}r)}=\liminf_{r+\infty}\frac{r^2}{6e^{4\Delta}r^2}=\frac{e^{-4\Delta}}{6}>0.$$

This implies that (3.4) *holds. Then we conclude ISS for system* (3.20) *by Theorem 12.*

Nevertheless, Theorem 13 does not allow to conclude ISS for this above example. It is obvious from the above that V *defined in relation* (3.21) *satisfies the conditions* (3.15) *and* (3.16). *We assume by contradiction that condition* (3.17) *also holds. Then, there exist* $\varepsilon>0$ *such that for all* $\phi\in\mathcal{X}^n$.

$$\alpha(Q(\phi(0)))=\frac{1}{4}|\phi(0)|^4\ge\varepsilon V_2(\phi(0))=\varepsilon|\phi_2(0)|^{12}$$

When we consider the set of $\phi\in\mathcal{X}^n$ *such that* $\phi=(0,\phi_2)$ *and* $\phi_2\neq 0$, *we get*

$$\frac{1}{4}|\phi_2(0)|^4\ge\varepsilon|\phi_2(0)|^{12}$$

This implies that

$$|\phi_2(0)|^8\le\frac{1}{4\varepsilon},$$

and then

$$|\phi_2(0)|\le\left(\frac{1}{4\varepsilon}\right)^{\frac{1}{8}}. \tag{3.23}$$

In particular, when we consider ϕ_2 *such that* $\phi_2(0)=(1+\frac{1}{4\varepsilon})^{\frac{1}{8}}$, (3.23) *becomes:*

$$1+\frac{1}{4\varepsilon}\le\frac{1}{4\varepsilon}. \tag{3.24}$$

This is a contradiction. Thus, we cannot apply Theorem 13 here to conclude ISS.

The following example shows that Theorems 12 and 13 that we have established are not a consequence of Theorem 11. This example gives us a system where our Theorems 12 and 13 allow to ensure the ISS but not Theorem 11.

Example 2. *Consider the following system:*

$$\dot{x}(t) = -2x(t)^3 + x(t)x(t-\Delta)^2 + x(t)u(t) \tag{3.25}$$

By considering the LKF V *:*

$$V(\phi) = \phi(0)^2 + \int_{-\Delta}^{0} \phi(s)^4 ds$$

we have:

$$\begin{aligned} D^+V(\phi, f(\phi, v)) &= -4\phi(0)^4 + 2\phi(0)^2\phi(-\Delta)^2 + 2\phi(0)^2 v + \phi(0)^4 - \phi(-\Delta)^4 \\ &= -3\phi(0)^4 + 2\phi(0)^2\phi(-\Delta)^2 + 2\phi(0)^2 v - \phi(-\Delta)^4 \\ &\leq -3\phi(0)^4 + \phi(0)^4 + \phi(-\Delta)^4 + \phi(0)^4 + v^2 - \phi(-\Delta)^4 \\ &\leq -\phi(0)^4 + v^2 \\ &\leq -\alpha(Q(\phi(0))) + \gamma(|v|). \end{aligned}$$

where $\alpha(s) = \gamma(s) = s^2$, *and* $Q(x) = x^2$ *for all* $s \geq 0,\ x \in \mathbb{R}$.
And then, we get

$$\begin{aligned} \nabla Q(\phi(0)) f(\phi, v) &= -4\phi(0)^4 + 2\phi(0)^2\phi(-\Delta)^2 + 2\phi(0)^2 v \\ &\leq -4\phi(0)^4 + \phi(0)^4 + \phi(-\Delta)^4 + \phi(0)^4 + v^2 \\ &\leq -2\phi(0)^2 + \phi(-\Delta)^4 + v^2 \\ &\leq \phi(-\Delta)^4 + v^2 \\ &\leq \left(\max_{\tau \in [-\Delta, 0]} |\phi(\tau)|^2 \right)^2 + v^2 \\ &= \left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau)) \right)^2 + v^2 \\ &\leq \sigma \left(\max_{\tau \in [-\Delta, 0]} Q(\phi(\tau)) \right) + \gamma(|v|) \end{aligned}$$

with $\sigma(s) = s^2$ *for all* $s \geq 0$. *So,*

$$\lim_{r \to +\infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} = \lim_{r \to +\infty} \frac{r^2}{r^2 e^{4\Delta}} = e^{-4\Delta} > 0.$$

And by Theorem 12, we can conclude that system (3.25) *is ISS.*

But, due to the term $-2x^3$ *which is in the dynamic of system* (3.25), *logically it cannot be exp-ISS and Theorem 11 cannot be applied.*

Futhermore, we can remark that $\alpha(Q(\phi(0))) = V_2(\phi(0)) = |\phi(0)|^4$. *Then* (3.17) *holds with* $\varepsilon = 1$, *and we have ISS by Theorem 13 too.*

3.3 Proofs

3.3.1 Proof of Lemma 2

By assumption $\liminf_{s\to\infty} \frac{f(s)}{g(s)} > 0$. So it holds that

$$\exists \eta, \varepsilon > 0 : \ \forall s > \eta, \ \frac{f(s)}{g(s)} > \varepsilon. \tag{3.26}$$

We consider the following two cases:

- Case 1: $\varepsilon \geq \frac{f(\eta)}{g(\eta)}$.
  Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as:

  $$\omega(s) := \begin{cases} \frac{f(s)}{g(\eta)} & \text{if } s \leq \eta \\ \frac{f(\eta)}{g(\eta)} & \text{if } s > \eta \end{cases}$$

  Then we have that

  (a) $\lim_{s\to\eta^-} \omega(s) = \lim_{s\to\eta^+} \omega(s) = \omega(\eta) = \frac{f(\eta)}{g(\eta)}$. Hence, ω is continuous on $\mathbb{R}_+$.

  (b) $\forall \ s \geq 0, \ |\omega(s)| \leq \frac{f(\eta)}{g(\eta)}$. Hence ω is bounded.

  (c) ω is non-decreasing, because $f \in \mathcal{K}_\infty$.

  (d) ω is positive definite

  (e) $\forall \ s \geq 0, \ \omega(s) \leq \frac{f(s)}{g(s)}$. Indeed, since $\varepsilon \geq \frac{f(\eta)}{g(\eta)}$, we get from (3.26) that, whenever $s > \eta$,

  $$\omega(s) = \frac{f(\eta)}{g(\eta)} \leq \varepsilon \leq \frac{f(s)}{g(s)}.$$

  And, when $0 \leq s \leq \eta$, we have $g(s) \leq g(\eta)$ since $g \in \mathcal{K}_\infty$, so $\omega(s) = \frac{f(s)}{g(\eta)} \leq \frac{f(s)}{g(s)}$.

- Case 2: $\varepsilon < \frac{f(\eta)}{g(\eta)}$.
  Then, we define $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ as

  $$\omega(s) := \begin{cases} \frac{f(s)}{g(\eta)} & \text{if } s < s_0 \\ \varepsilon & \text{if } s \geq s_0 \end{cases}$$

  where $s_0 = f^{-1}(\varepsilon g(\eta))$ (f is invertible, so f^{-1} is uniquely defined and s_0 too). Reasoning as in Case 1, ω is continuous, positive definite, bounded and non-decreasing. In addition, we have

  - If $s > \eta, \ \omega(s) = \varepsilon \leq \frac{f(s)}{g(s)}$ by (3.26).
  - If $s_0 \leq s \leq \eta, \ \omega(s) = \varepsilon < \frac{f(s)}{g(\eta)} \leq \frac{f(s)}{g(s)}$ by using the fact that $\varepsilon < \frac{f(\eta)}{g(\eta)}$ and that $f, g \in \mathcal{K}_\infty$.
  - If $0 \leq s < s_0, \ \omega(s) = \frac{f(s)}{g(\eta)} \leq \frac{f(s)}{g(s)}$ since $g \in \mathcal{K}_\infty$.

Then, in all cases, there exist a function ω continuous, bounded and non-decreasing such that $0 \leq \omega(s) \leq \frac{f(s)}{g(s)}$.

3.3.2 Proof of Theorem 12

Consider the functional defined for all $\phi \in \mathcal{X}^n$ as

$$V_0(\phi) = \max_{\tau \in [-\Delta,0]} e^{2\tau} Q(\phi(\tau)).$$

Then, by Lemma 1, V_0 is Lipschitz on bounded sets. Moreover, it holds that

$$e^{-2\Delta} \max_{\tau \in [-\Delta,0]} Q(\phi(\tau)) \leq V_0(\phi) \leq \max_{\tau \in [-\Delta,0]} Q(\phi(\tau)). \tag{3.27}$$

In view of (3.4), it holds that

$$\liminf_{r \to \infty} \frac{\alpha(r)}{\sigma(e^{2\Delta} r)} > 0,$$

which in turn implies that

$$\liminf_{s \to \infty} \frac{\alpha(s)}{2\sigma(e^{2\Delta} s)} > 0.$$

By Lemma 2, it follows that there exists a continuous, positive definite, nondecreasing and bounded function ε_0 such that

$$\frac{\alpha(s)}{2\sigma(e^{2\Delta} s)} \geq \varepsilon_0(s), \quad \forall s \geq 0. \tag{3.28}$$

Consider the continuously differentiable $\mathcal{K}_\infty$ function ε defined as

$$\varepsilon(s) := \int_0^s \varepsilon_0(r) dr, \quad \forall s \geq 0,$$

and the functional $W : \mathcal{X}^n \to \mathbb{R}_+$ defined as

$$W(\phi) := V(\phi) + \varepsilon(V_0(\phi)), \quad \forall \phi \in \mathcal{X}^n.$$

Since ε is continuously differentiable, W is Lipschitz on bounded sets. By (3.1) and (3.27), we have that

$$\varepsilon \left(e^{-2\Delta} \max_{\tau \in [-\Delta,0]} Q(\phi(\tau)) \right) \leq W(\phi) \leq \bar{\alpha}(\|\phi\|) + \varepsilon \left(\max_{\tau \in [-\Delta,0]} Q(\phi(\tau)) \right). \tag{3.29}$$

In addition, due to the fact that Q is continuously differentiable, positive definite and radially unbounded, there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq Q(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n,$$

which ensures in particular that

$$\alpha_1(\|\phi\|) \leq \max_{\tau \in [-\Delta,0]} Q(\phi(\tau)) \leq \alpha_2(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n. \tag{3.30}$$

Combining (3.29) and (3.30), we get that

$$\varepsilon \left(e^{-2\Delta} \alpha_1(\|\phi\|) \right) \leq W(\phi) \leq \bar{\alpha}(\|\phi\|) + \varepsilon \left(\alpha_2(\|\phi\|) \right). \tag{3.31}$$

From Lemma 1, V_0 satisfies

$$V_0(\phi) > Q(\phi(0)) \quad \Rightarrow \quad D^+V_0(\phi, f(\phi, v)) \leq -2V_0(\phi) \tag{3.32}$$
$$V_0(\phi) = Q(\phi(0)) \quad \Rightarrow \quad D^+V_0(\phi, f(\phi, v)) \leq \max\{-2V_0(\phi), \nabla Q(\phi(0))f(\phi, v)\}. \tag{3.33}$$

In addition, using (3.2) and Lemma 3, it holds that

$$D^+W(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|) + \varepsilon_0(V_0)D^+V_0(\phi, f(\phi, v)). \tag{3.34}$$

In view of (3.32)-(3.33), we consider two cases.

- Case 1: $V_0(\phi) > Q(\phi(0))$.
  Then we have from (3.32) and (3.34) that

  $$\begin{aligned} D^+W(\phi, f(\phi, v)) &\leq -\alpha(Q(\phi(0))) + \gamma(|v|) - 2\varepsilon_0(V_0(\phi))V_0(\phi) \\ &\leq -2\varepsilon_0(V_0(\phi))V_0(\phi) + \gamma(|v|). \end{aligned}$$

  Using (3.27) and (3.30), it follows that

  $$\begin{aligned} D^+W(\phi, f(\phi, v)) &\leq -2\varepsilon_0 \left(e^{-2\Delta} \max_{\tau\in[-\Delta,0]} Q(\phi(\tau))\right) e^{-2\Delta} \max_{\tau\in[-\Delta,0]} Q(\phi(\tau)) + \gamma(|v|) \\ &\leq -2\varepsilon_0 \left(e^{-2\Delta}\alpha_1(\|\phi\|)\right) e^{-2\Delta}\alpha_1(\|\phi\|) + \gamma(|v|). \end{aligned} \tag{3.35}$$

- Case 2: $V_0(\phi) = Q(\phi(0))$.
  Then we have from (3.33) and (3.34) that

  $$D^+W(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0)))+\gamma(|v|)+\varepsilon_0(V_0(\phi)) \max\{-2V_0(\phi), \nabla Q(\phi(0))f(\phi, v)\}.$$

  - Subcase 2.1: $-2V_0(\phi) \geq \nabla Q(\phi(0))f(\phi, v)$.
    Then, proceeding as in Case 1, we get that

    $$\begin{aligned} D^+W(\phi, f(\phi, v)) &\leq -\alpha(Q(\phi(0))) + \gamma(|v|) - 2\varepsilon_0(V_0(\phi))V_0(\phi, f(\phi, v)) \\ &\leq -2\varepsilon_0(e^{-2\Delta}\alpha_1(\|\phi\|))e^{-2\Delta}\alpha_1(\|\phi\|) + \gamma(|v|). \end{aligned} \tag{3.36}$$

  - Subcase 2.2: $-2V_0(\phi) < \nabla Q(\phi(0))f(\phi, v)$.
    Then we make use of (3.3) to get that

    $$\begin{aligned} D^+W(\phi, f(\phi, v)) &\leq -\alpha(Q(\phi(0))) + \gamma(|v|) + \varepsilon_0(V_0(\phi))\nabla Q(\phi(0))f(\phi, v) \\ &\leq -\alpha(V_0(\phi)) + \varepsilon_0(V_0(\phi))\sigma \left(\max_{\tau\in[-\Delta,0]} Q(\phi(\tau))\right) \\ &\quad + (\varepsilon_0(V_0(\phi)) + 1)\, \gamma(|v|). \end{aligned}$$

    Since ε_0 is bounded, there exists $\bar{\varepsilon}_0 > 0$ such that $\varepsilon_0(s) \leq \bar{\varepsilon}_0$ for all $s \geq 0$. Using also (3.27) and the fact that σ is non-decreasing by assumption, we obtain that

    $$D^+W(\phi, f(\phi, v)) \leq -\alpha(V_0(\phi)) + \varepsilon_0(V_0(\phi))\sigma\left(e^{2\Delta}V_0(\phi)\right) + (\bar{\varepsilon}_0 + 1)\gamma(|v|). \tag{3.37}$$

In view of (3.28), it holds that

$$\varepsilon_0(V_0(\phi)) \leq \frac{\alpha(V_0(\phi))}{2\sigma\left(e^{2\Delta}V_0(\phi)\right)}$$

Using this, (3.27) and (3.30), we get from (3.37) that

$$\begin{aligned}
D^+W(\phi, f(\phi,v)) &\leq -\frac{1}{2}\alpha(V_0(\phi)) + (\bar{\varepsilon}_0+1)\gamma(|v|) \\
&\leq -\frac{1}{2}\alpha(e^{-2\Delta}\max_{\tau\in[-\Delta,0]} Q(\phi(\tau))) + (\bar{\varepsilon}_0+1)\gamma(|v|) \\
&\leq -\frac{1}{2}\alpha(e^{-2\Delta}\alpha_1(\|\phi\|)) + (\bar{\varepsilon}_0+1)\gamma(|v|). \qquad (3.38)
\end{aligned}$$

Combining (3.35), (3.36) and (3.38), we finally obtain that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$D^+W(\phi, f(\phi,v)) \leq -\tilde{\alpha}(\|\phi\|) + \tilde{\gamma}(|v|) \qquad (3.39)$$

where

$$\begin{aligned}
\tilde{\alpha}(s) &:= \min\left\{\varepsilon_0\left(e^{-2\Delta}\alpha_1(s)\right)e^{-2\Delta}\alpha_1(s), \alpha(e^{-2\Delta}\alpha_1(s))\right\} \\
\tilde{\gamma}(s) &:= (\bar{\varepsilon}_0+1)\gamma(s).
\end{aligned}$$

It can be checked that $\tilde{\alpha}, \tilde{\gamma} \in \mathcal{K}_\infty$. In view of (3.31) and (3.39), we conclude that W is a coercive LKF with history-wise dissipation, so the system (1.1) is ISS by Theorem 8.

3.3.3 Proof of Corollary 2

We start by showing that if $\alpha \in \mathcal{K}_\infty \cap C^1$ and $\alpha'(0) = 0$, then the function $Q : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ defined as $Q(x) := \alpha(|x|)$ for all $x \in \mathbb{R}^n$ is C^1. Indeed, it is obvious to see that Q is C^1 on $\mathbb{R}^n \setminus \{0\}$, so it is enough to show that Q is differentiable at 0 and that its gradient is continuous at 0. We have

$$\lim_{|x|\to 0}\frac{Q(x)-Q(0)}{|x|} = \lim_{|x|\to 0}\frac{\alpha(|x|)-\alpha(0)}{|x|} = \alpha'(0) = 0.$$

This shows that Q is differentiable in 0. Moreover

$$\left|\lim_{|x|\to 0^+}\nabla Q(x)\right| \leq \lim_{|x|\to 0^+}\left|\alpha'(|x|)\frac{x^T}{|x|}\right| \leq \lim_{|x|\to 0^+}|\alpha'(|x|)| = 0,$$

thus showing that ∇Q is continuous at zero. Hence, Q is indeed C^1.

Since Q is continuously differentiable, we can apply Theorem 12. More precisely, it holds from (3.10) and (3.11) that

$$\begin{aligned}
D^+V(\phi, f(\phi,v)) &\leq -Q(\phi(0)) + \gamma(|v|) \\
\nabla Q(\phi(0))f(\phi,v) &\leq \sigma\circ\alpha^{-1}\left(\max_{\tau\in[-\Delta,0]} Q(\phi(\tau))\right) + \gamma(|v|).
\end{aligned}$$

Consequently, ISS holds from Corollary 1 provided that

$$\liminf_{r\to\infty}\frac{r}{\sigma\circ\alpha^{-1}(r)} > 0,$$

which is ensured by (3.12).

3.3.4 Proof of Theorem 13

Since W is an LKF, V_1, V_2 are Lyapunov functions. And then, by [Kellett, 2014, Definition 6], there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}_\infty$ such that for all $x \geq 0$:

$$\alpha_1(|x|) \leq V_1(x) \leq \alpha_2(|x|) \tag{3.40}$$

$$\beta_1(|x|) \leq V_2(x) \leq \beta_2(|x|) \tag{3.41}$$

Then, for all $x \geq 0$

$$V_2(x) \geq \beta_1 \circ \alpha_2^{-1}(V_1(x)) \tag{3.42}$$

By differentiating (3.15) and (3.18), we have

$$D^+W(\phi, f(\phi, v)) = D^+V_1(\phi(0)) + V_2(\phi(0)) - V_2(\phi(-\Delta)) \tag{3.43}$$

$$\begin{aligned} D^+V(\phi, f(\phi, v)) =& D^+V_1(\phi(0)) + pV_2(\phi(0)) - pe^{-\Delta c}V_2(\phi(-\Delta)) \\ & - pc\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds \end{aligned} \tag{3.44}$$

Combining (3.44) and (3.43), we get

$$\begin{aligned} D^+V(\phi, f(\phi, v)) =& D^+W(\phi, f(\phi, v)) - (1-p)V_2(\phi(0)) \\ & - (pe^{-\Delta c} - 1)V_2(\phi(-\Delta)) - pc\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds \end{aligned}$$

By (3.16), it becomes

$$\begin{aligned} D^+V(\phi, f(\phi, v)) \leq & - \alpha(Q(\phi(0))) - (1-p)V_2(\phi(0)) - (pe^{-\Delta c} - 1)V_2(\phi(-\Delta)) \\ & - pc\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds + \gamma(|v|) \end{aligned}$$

And due to assumption (3.17), we get

$$\begin{aligned} D^+V(\phi, f(\phi, v)) \leq & - (\varepsilon + 1 - p)\, V_2(\phi(0)) - (pe^{-\Delta c} - 1)V_2(\phi(-\Delta)) \\ & - pc\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds + \gamma(|v|) \end{aligned}$$

Let's consider $c > 0$ such that $e^{\Delta c} < 1 + \varepsilon$. Thus, for any $e^{\Delta c} < p < 1 + \varepsilon$, we have

$$\begin{cases} \varepsilon + 1 - p > 0 \\ pe^{-\Delta c} - 1 > 0. \end{cases}$$

And then,

$$D^+V(\phi, f(\phi, v)) \leq -\rho V_2(\phi(0)) - pc\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds + \gamma(|v|), \quad \rho = \varepsilon + 1 - p \tag{3.45}$$

By (3.42), (3.45) becomes

$$\begin{aligned} D^+V(\phi, f(\phi, v)) &\leq -\rho\beta_1 \circ \alpha_2^{-1}(V_1(\phi(0))) - pc\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds + \gamma(|v|) \\ &\leq -c\left(\lambda(V_1(\phi(0))) + p\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds\right) + \gamma(|v|) \quad \text{where } \lambda = \frac{\rho}{c}\beta_1 \circ \alpha_2^{-1} \\ &\leq -c\left(\bar{\lambda}(V_1(\phi(0))) + \bar{\lambda}\left(p\int_{-\Delta}^{0} e^{cs}V_2(\phi(s))ds\right)\right) + \gamma(|v|) \end{aligned}$$

where $\bar{\lambda}(s) = \min\{\lambda(s), s\}$. It is obvious to see that $\bar{\lambda} \in \mathcal{K}_\infty$. And then by Lemma 4, for all $s, t \geq 0$, $\bar{\lambda}(2s) + \bar{\lambda}(2t) \geq \bar{\lambda}(s+t)$. Let's call $\tilde{\lambda}$, the $\mathcal{K}_\infty$ function such that

$$\tilde{\lambda}(s) = \bar{\lambda}\left(\frac{1}{2}s\right)$$

Thus, we get

$$D^+V(\phi, f(\phi, v)) \leq -c\tilde{\lambda}(V) + \gamma(|v|).$$

Hence, by Theorem 8, the system (1.1) is ISS.

Conclusion and perspectives

In this report, we have recalled all the essential notions on global asymptotic stability (GAS), global exponential stability (GES) and ISS of time-delay systems. We have noted that fundamental approach to study these notions is the Lyapunov approach. The latter consists in exhibiting a functional which decreases along the solutions of system. In the case of delay systems, this functional is called Lyapunov-Krasovskii functional (LKF) and it dissipates in different forms (point-wisely, history-wise, and LKF-wise). It turns out that all these forms are equivalent for the global asymptotic stability. But for the global exponential stability and the ISS, the only results that exist use either an LKF with strict LKF-dissipation or an LKF with history-wise dissipation. As for the characterization with point-wise dissipation, it has been conjectured in [Chaillet et al., 2023b] that it would also ensure the ISS and GES. There have already been results that propose sufficient conditions for these conjectures to be true. But these conditions are still restrictive. Thus, in this report we have proposed results on ISS that extend the previous ones to a broader class of delay systems. Concretely, we have proposed sufficient conditions to guarantee the ISS with point-wise dissipation, and these conditions apply to some classes of systems to which the past results do not apply. Finally, it must be said that the conjectures are not yet proven and this is precisely what we will investigate further in the future. If we cannot prove that they are true, we will aim to construct counterexamples which show that they may be false.

Other future investigations

In future, for my Ph.D, I will also work on the ISS of some PDEs systems.

- The first problem concerns evolution systems modeled with PDEs. Many exponential stability results have been recently shown with a powerful method called Fredholm backstepping [Coron and Lü, 2014], [Coron et al., 2018], [Coron et al., 2022]. We would like to know whether the Lyapunov functionals that can be deduced from this method could also ensure the ISS property, and for which type of perturbations.

- The second problem combines both PDEs and time-delays. For PDE systems, the exponential stability results obtained with so-called basic quadratic Lyapunov functions also imply in many cases the ISS of the system [Bastin et al., 2021]. However, when there are time delays in the application of the control or in the dynamics such a positive general result is not known. In fact, it was even shown that in some cases the system is not robust to time-delays, hence ISS is lost [Auriol et al., 2018]. We would like to find a general condition ensuring the ISS in this framework.

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