Getting LKF-wise dissipation to ensure ISS for Time-Delay Systems

École des Ponts ParisTech

Epiphane Loko, Antoine Chaillet, Yuan Wang, Iasson Karafyllis, Pierdomenico Pepe

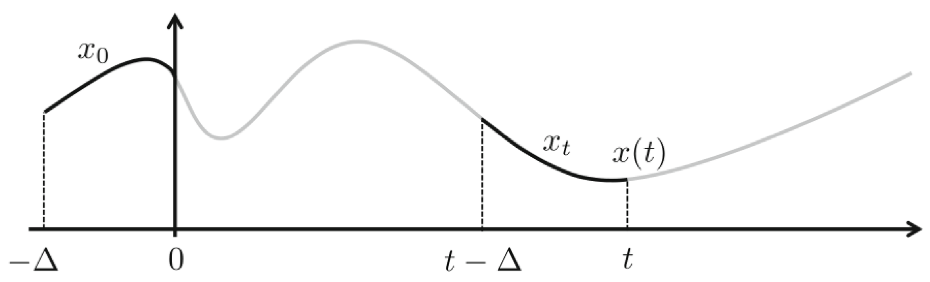
L2S Laboratoire Signaux & Systèmes

Abstract

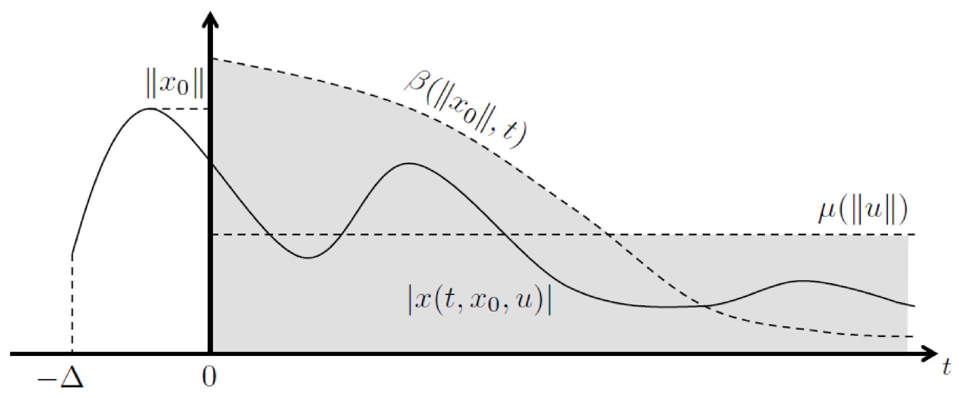
For time-delay systems, we want to make the Lyapunov ISS analysis like in finite dimension: $\dot{x}(t) = f(x(t), u(t))$.

Time-delay systems & LKF

• **Time-delay systems** (TDS):
$\dot{x}(t) = f(x_t, u(t))$, $x_t : [-\Delta, 0] \ni s \mapsto x(t+s) \in \mathbb{R}^n$.



• **Input-to-state stability** (ISS):
$|x(t, x_0, u)| \leq \max\{\beta(||x_0||, t), \mu(|u(t)|)\}$; $||x_0|| = \max_{s\in[-\Delta,0]} |x_0(s)|$.



• **Lyapunov-Krasovskii functional** (LKF): V Lipschitz on bounded sets and

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(||x_t||).$$

Theorem 1(Karafyllis et al.,2008): ISS of TDS $\Leftrightarrow \exists$ LKF V :

$$D^+V \leq \boxed{-\alpha(V(x_t))} + \gamma(|u(t)|). \tag{1}$$

↑↑↑

LKF-wise dissipation: sometimes hard to obtain

Conjecture(Chaillet et al.,2017) ISS of TDS $\Leftrightarrow \exists$ LKF V with

$$D^+V \leq \boxed{-\alpha(|x(t)|)} + \gamma(|u(t)|). \tag{2}$$

↑↑↑

point-wise dissipation: easy to obtain

Point-wise dissipation

The adding exponential trick

- When it works:

$$V(x_t) = w_1(x(t)) + \int_{-\Delta}^{0} w_2(x_t(s))ds. \tag{3}$$

Proposition 1(Loko et al.,2024b): If the LKF (3) satisfies (2) with $w_2(z) \leq \varepsilon\alpha(|z|)$, then

$$W(x_t) = w_1(x(t)) + \int_{-\Delta}^{0} ke^{cs} w_2(x_t(s))ds \tag{4}$$

is an ISS LKF with LKF-wise dissipation and the TDS is ISS.

- When it does not work

Proposition 2(Loko et al.,2024b): Consider the following 1D TDS:

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}. \tag{5}$$

  * The TDS (5) is ISS.
  * The LKF V in (3) with $w_2(z) = 4w_1(z) = z^4$ dissipates point-wisely.
  * For any $k, c > 0$, the corresponding LKF W (4) does not dissipate LKF-wise.

ISS with point-wise dissipation

Theorem 2(Loko et al., 2024b): Given a LKF V dissipating as

$$D^+V \leq -\alpha(Q(x(t))) + \gamma(|u(t)|)$$

with Q smooth and positive definite function. Easy to show that

$$\dot{Q}(x(t)) \leq \sigma(||x_t||) + \gamma(|u(t)|).$$

Then the considered TDS is ISS provided that

$$\liminf_{s\to+\infty} \alpha(s)/\sigma(se^{\Delta}) > 0.$$

Example 1

$$\dot{x}(t) = -2x(t)^3 + x(t)x(t-\Delta)^2 + x(t)u(t). \tag{6}$$

- Using Theorem 2 or Proposition 1, we are able to show the ISS of TDS (6).
- Existing ISS results like in (Chaillet et al., 2023) and in (Orlowski et al., 2022) do not apply.

Example 2

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + x_2(t-\Delta)^3 + u(t) \\ \dot{x}_2(t) = -x_2(t) - x_2(t)^9 \end{cases} \tag{7}$$

- The TDS (7) is ISS based on Theorem 2.
- ISS result in (Chaillet et al., 2017) does not apply.

Coercive LKF building

Theorem 3 (Loko et al., 2024a): For any function Q satisfying Razumikhin condition presented in (Teel, 1998) or Halanay condition presented in (Halanay, 1966), the LKF V defined as

$$V(x_t) = \max_{s\in[-\Delta,0]} e^{cs} Q(x_t(s))$$

is a coercive LKF with LKF-wise dissipation for the considered TDS.

Application: Chemical reactor model

$$\partial_t x_c(t,z) + c\partial_z x_c(t,z) = -\xi x_c(t,z) + \xi x(t)$$
$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu\int_0^1 x_c(t,z)dz + u(t).$$

- ISS requirement is improved for the system.
- Unlike the several existing stability analysis, we used the sup norm.

Conclusion & Perspectives

- The conjecture is still an open question.
- Improve our Razumikhin and Halanay conditions.
- Look for LKF from vector version of Halanay.

References

SCAN ME